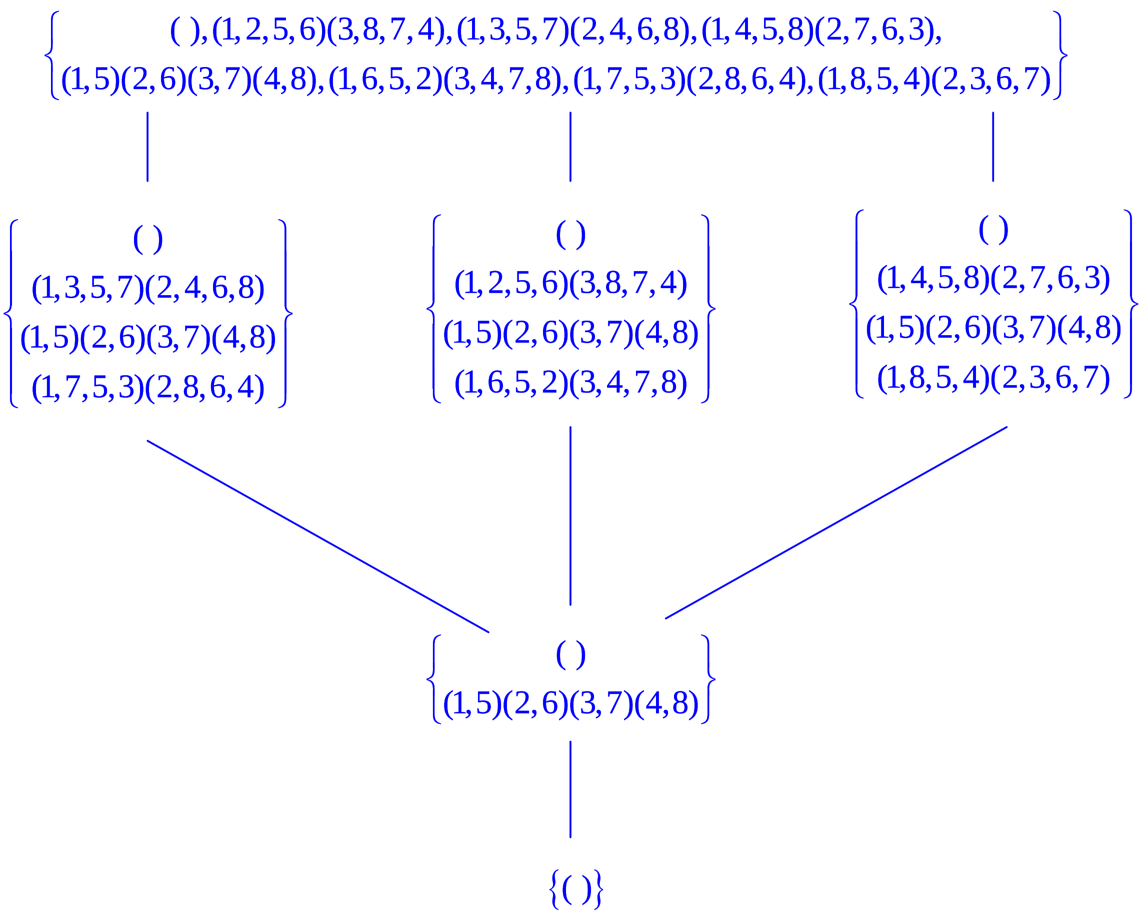
# WHY NORMAL SUBGROUPS ARE SO IMPORTANT

Recall that a subgroup $H$ of a group $G$ is called normal or self-conjugate if for every element $b$ in $G$ we have that $b^{-1}Hb = H$, and since we are considering all elements in $G$, this also means that $bHb^{-1} = H$. To get this form, just start with $b^{-1}$ and conclude that $(b^{-1})^{-1}Hb^{-1} = bHb^{-1} = H$. And now, the reason why normal subgroups are so important is because their cosets, either the right or the left, will also give us a group, and the order of this group is $\frac{|G|}{|H|}$, the same as the number of cosets.

Along with this claim that we have group, we need to also explain how multiplication is done, and to do this we will use the quaternion group since it is known that every subgroup of that group is normal. Thus, let $Q$ = { (), (1,2,5,6)(3,8,7,4), (1,3,5,7)(2,4,6,8), (1,4,5,8)(2,7,6,3), (1,5)(2,6)(3,7)(4,8), (1,6,5,2)(3,4,7,8), (1,7,5,3)(2,8,6,4), (1,8,5,4)(2,3,6,7)} and recall the subgroup lattice we found earlier for this group.



From this group, let's let $H=\left\{\begin{matrix}(\ )\\(1,5)(2,6)(3,7)(4,8)\end{matrix}\right\}$. Then this is subgroup of order 2 that lies within a group of order 8, and so the number of distinct right (or left) cosets we can form is $\frac{|G|}{|H|}=\frac{8}{2}=4$. We will focus on the right cosets since when we learn how to use the *GAP* program (*Groups, Algorithms, and Programming*), that program will work primarily with right instead of left cosets. Nonetheless, everything we will do can just as easily be done with the left cosets as well.

To find another coset of *H* (different from *H* itself), just pick an element in *Q* that is not already in *H*, and form the corresponding right coset. For example,

$$H(1,2,5,6)(3,8,7,4)=\left\{\begin{matrix}(\ )\\(1,5)(2,6)(3,7)(4,8)\end{matrix}\right\}(1,2,5,6)(3,8,7,4)$$
$$=\left\{\begin{matrix}(\ )(1,2,5,6)(3,8,7,4)\\(1,5)(2,6)(3,7)(4,8)(1,2,5,6)(3,8,7,4)\end{matrix}\right\}=\left\{\begin{matrix}(1,2,5,6)(3,8,7,4)\\(1,6,5,2)(3,4,7,8)\end{matrix}\right\}$$

Now pick any element that is not in the first two right cosets, and repeat the process.

$$H(1,3,5,7)(2,4,6,8)=\left\{\begin{matrix}(\ )\\(1,5)(2,6)(3,7)(4,8)\end{matrix}\right\}(1,3,5,7)(2,4,6,8)$$
$$=\left\{\begin{matrix}(\ )(1,3,5,7)(2,4,6,8)\\(1,5)(2,6)(3,7)(4,8)(1,3,5,7)(2,4,6,8)\end{matrix}\right\}=\left\{\begin{matrix}(1,3,5,7)(2,4,6,8)\\(1,7,5,3)(2,8,6,4)\end{matrix}\right\}$$

We've now found three right cosets which we can denote by *H*, $H(1,2,5,6)(3,8,7,4)$, and $H(1,3,5,7)(2,4,6,8)$. We need just one more, and so we look for an element that is still not contained in one of the cosets we've constructed, we can use that element to construct our final right coset. In particular, let's construct $H(1,,8,5,4)(2,3,6,7)$.

$$H(1,8,5,4)(2,3,6,7)=\left\{\begin{matrix}(\ )\\(1,5)(2,6)(3,7)(4,8)\end{matrix}\right\}(1,8,5,4)(2,3,6,7)$$
$$=\left\{\begin{matrix}(\ )(1,8,5,4)(2,3,6,7)\\(1,5)(2,6)(3,7)(4,8)(1,8,5,4)(2,3,6,7)\end{matrix}\right\}=\left\{\begin{matrix}(1,8,5,4)(2,3,6,7)\\(1,4,5,8)(2,7,6,3)\end{matrix}\right\}$$

And now we have completed constructing our four right cosets. Notice that each coset contains the same number of elements and that all the elements in the group fall into one coset or another. Among other things, this means that the product of the number of cosets with the number of elements in a coset gives use back the number of elements in a group. Or, another way to say it, the number of elements in our subgroup *H* will always be a divisor of the number of elements in the group.

$$H = \begin{Bmatrix} (\,) \\ (1,5)(2,6)(3,7)(4,8) \end{Bmatrix}$$

$$H(1,2,5,6)(3,8,7,4) = \begin{Bmatrix} (1,2,5,6)(3,8,7,4) \\ (1,6,5,2)(3,4,7,8) \end{Bmatrix}$$

$$H(1,3,5,7)(2,4,6,8) = \begin{Bmatrix} (1,3,5,7)(2,4,6,8) \\ (1,7,5,3)(2,8,6,4) \end{Bmatrix}$$

$$H(1,8,5,4)(2,3,6,7) = \begin{Bmatrix} (1,8,5,4)(2,3,6,7) \\ (1,4,5,8)(2,7,6,3) \end{Bmatrix}$$

Now let's make the following point. Notice that the coset $H(1,2,5,6)(3,8,7,4) = \begin{Bmatrix} (1,2,5,6)(3,8,7,4) \\ (1,6,5,2)(3,4,7,8) \end{Bmatrix}$ contains two elements, and we have represented this right coset as *H* times the permutation $(1,2,5,6)(3,8,7,4)$. However, we could just as easily have represented this coset as *H* times the second permutation in the coset, $(1,6,5,2)(3,4,7,8)$. This sort of thing will always happen. We can always pick any other element out of our coset, multiple our subgroup *H* by it on the appropriate side, and we'll always get back the same coset. To illustrate this, just observe that,

$$H(1,6,5,2)(3,4,7,8) = \begin{Bmatrix} (\,) \\ (1,5)(2,6)(3,7)(4,8) \end{Bmatrix}(1,6,5,2)(3,4,7,8)$$
$$= \begin{Bmatrix} (\,)(1,6,5,2)(3,4,7,8) \\ (1,5)(2,6)(3,7)(4,8)(1,6,5,2)(3,4,7,8) \end{Bmatrix} = \begin{Bmatrix} (1,6,5,2)(3,4,7,8) \\ (1,2,5,6)(3,8,7,4) \end{Bmatrix}.$$
$$= \begin{Bmatrix} (1,2,5,6)(3,8,7,4) \\ (1,6,5,2)(3,4,7,8) \end{Bmatrix} = H(1,2,5,6)(3,8,7,4)$$

Thus, when we are trying to represent a right coset as the product of our subgroup *H* and an element in that coset, it doesn't matter which element of the coset we pick. We'll always get the same result, and the same is true for left cosets, too.

We've stated that when our subgroup *H* is normal in our group, then the right (left) cosets form a group. To verify this, we need to state how we do multiplication in this group, and it couldn't be simpler. If *H*a and *Hb* are right cosets, then $Ha \cdot Hb = Hab$. One thing we need to know, though, is that we will always get the same product no matter which representatives we pick from our cosets for *a* and *b*, and, no need to worry, we always will get the same result. And we'll illustrate this using our example cosets above.

First,

$$H(1,2,5,6)(3,8,7,4)\cdot H(1,3,5,7)(2,4,6,8) = H(1,2,5,6)(3,8,7,4)(1,3,5,7)(2,4,6,8)$$
$$= H(1,4,5,8)(2,7,6,3) = \begin{Bmatrix}(1,4,5,8)(2,7,6,3)\\(1,8,5,4)(2,3,6,7)\end{Bmatrix}.$$

And using different representatives from each coset, we get exactly the same result.

$$H(1,6,5,2)(3,4,7,8)\cdot H(1,7,5,3)(2,8,6,4) = H(1,6,5,2)(3,4,7,8)(1,7,5,3)(2,8,6,4)$$
$$= H(1,4,5,8)(2,7,6,3) = \begin{Bmatrix}(1,4,5,8)(2,7,6,3)\\(1,8,5,4)(2,3,6,7)\end{Bmatrix}$$

Thus, when our subgroup *H* is normal or self-conjugate, we get the same result for multiplication of cosets no matter which representatives we pick from our cosets to use in determining the multiplication. And this happens only when *H* is normal. This is a theorem that we'll prove much later on. Also, remember that when a subgroup is normal, its right cosets equal its left cosets, and so the group that is obtained is exactly the same regardless of whether we are using left or right cosets. And this group is traditionally called either a quotient group or a factor group.