DOC BENTON'S FORBIDDEN SECRETS OF MULTIVARIABLE CALCULUS

VOLUME II

Derivatives, Integrals, Vector Fields, Line Integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem

CHRISTOPHER P. BENTON, PHD

DEDICATED TO

שושן דודי

docbenton@docbenton.com

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INTRODUCTION

Why is this multivariable calculus book different from all other multivariable calculus books? Well, for one thing, it's free! Frankly, I'm tired of all those textbooks that cost a small fortune to purchase. Education should be affordable for everyone!

Another thing that makes this book different is that it is not really meant to be a textbook or to replace whatever book you are using in class. Instead, it's meant to be a story, the story of multivariable calculus, and like all good stories it has a beginning and an end and the various parts of the story are all interconnected with one another. That is one of the things I wanted to show here, how multivariable calculus is a grand adventure where tools developed early on become critical parts of the story that follows. Additionally, I've left out some topics (with regret) and expanded others (with glee!) simply to make the story flow a little better. I don't cover everything that one might cover in multivariable calculus, but I do cover a lot and I do enough to give you a good basis for what it's all about.

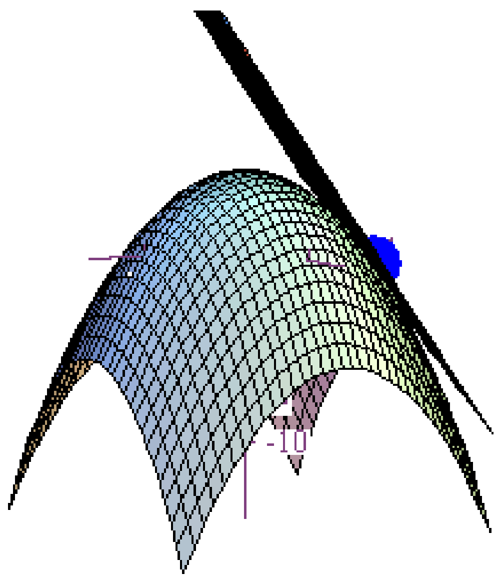
And lastly, this book is different because I've created a lot of online resources that are available for anyone to use for free! These include problem sets that also double as additional examples, graphers and calculators for functions of several variables that allow constructions and explorations that go far beyond anything that was available when I first took calculus in the early seventies, and PowerPoint slide presentations that go along with this book and sometimes even cover things that I haven't included in this book. To access this material, just go to www.docbenton.com, and look for links to my multivariable calculus courses and stuff. And above all, enjoy!

-Doc Benton

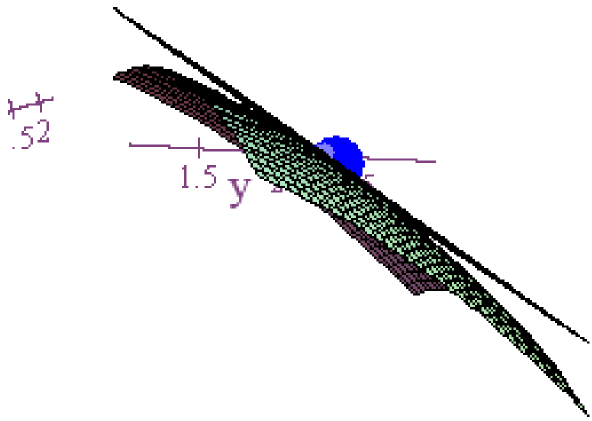
CHAPTER 7

DERIVATIVES

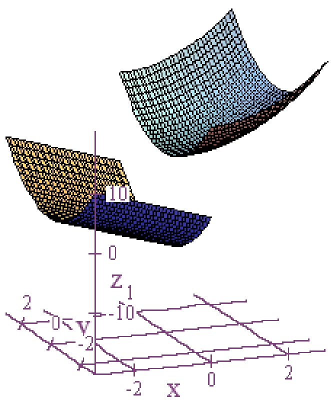
At last we have arrived! We're finally ready to look at the wide, wide world of derivatives of multivariable functions! And as you might suspect, just as graphing is richer and more complex in three dimensions, so will the concept of a derivative have more dimensions to it (pun intended!). In a nutshell, though, we will say that a function of several variables is differentiable at a point if we can define a nonvertical tangent plane at that point. For example, below is a graph of $z = f(x, y) = -x^2 - y^2 + 4$ with a tangent plane plotted at the point $(1, 2, -1)$.

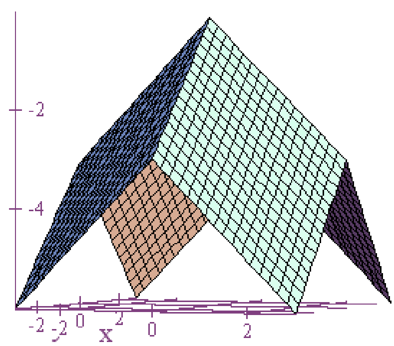


When we look at this tangent plane, we see that every single tangent line at the point $(1,2,-1)$ is contained in that plane, and we also realize that the tangent plane is a good approximation for the function when our input values are close to $x=1$ and $y=2$. Also, because we can define a tangent plane at this point, that means that as we zoom in on this point the graph of $z=f(x,y)=-x^2-y^2+4$ resembles the tangent plane more and more. That property is what we call local linearity, and you can easily see this property for many functions in two dimensions using a standard graphing calculator. Just take function such as $y=f(x)=x^2$, zoom in on any point on the graph, and your curve will resemble a straight line more and more as you zoom in. In other words, $y=f(x)=x^2$ is locally linear. Here is the graph of $z=f(x,y)=-x^2-y^2+4$ zoomed in around the point $(1,2,-1)$ so that you can see the local linearity starting to take effect.



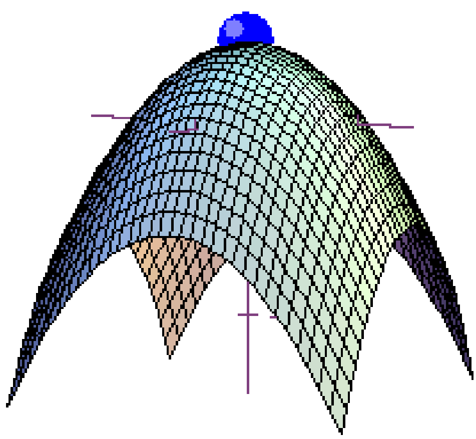
The appropriate question to ask now is what sorts of things would prevent a graph from being differentiable at a point? Normally, the same sorts of things that cause nondifferentiability back in two dimensions. In other words, breaks in continuity and sharp points. Here are a couple of examples.



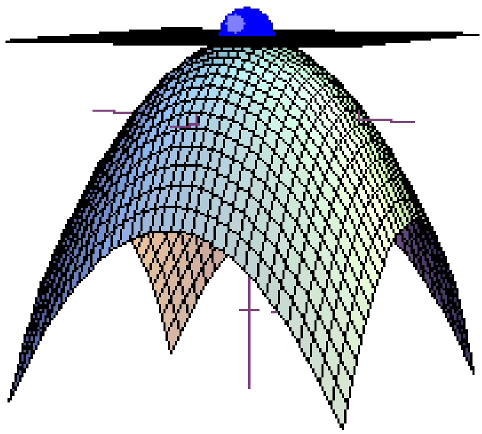


In the first graph, we're not going to be able to define tangent planes where the breaks in the graph occur, and in the second surface, we can't define tangent planes at the sharp edges or at the sharp point at the top of the graph. As you've seen before, derivatives require smoothness in order to be defined, and wherever a graph is not smooth, it's not going to be differentiable.

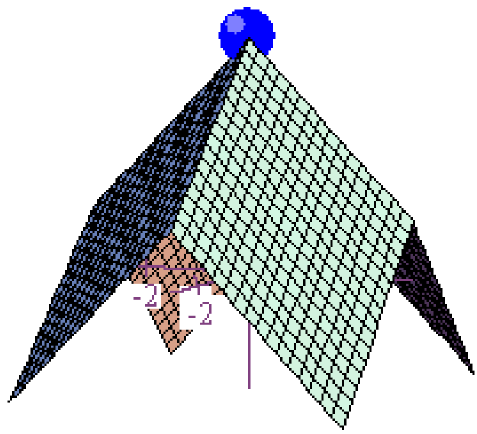
Now let's go back to the graph of $z = f(x,y) = -x^2 - y^2 + 4$. What's obvious here is that the point with coordinates $(0,0,4)$ represents the top of a hill. When that happens we call it a local maximum. Furthermore, if there are no other points or hilltops on the graph that are higher, then we also call it an absolute maximum.



If we now add the tangent plane to this point, then we can visually see something very important. Namely, that it's horizontal. This also means that that all the tangent lines in all directions have slope zero.



On the other hand, if we look at the graph of $z = -|x| - |y| + 4$, then we can see that it also has a local and absolute maximum at $(0,0,4)$, but no tangent plane exists at this point since it's a sharp point. This also tells us that tangent lines will generally fail to exist in some if not all directions when we can't define a tangent plane.



Back in first semester calculus you probably saw that if a function $y = f(x)$ had a local maximum or minimum at a point, then either the derivative was zero at that point or it was undefined. A similar criteria exists for functions of the form $z = f(x, y)$. If a local maximum or minimum exists for a function of this sort, then either the slopes of the tangent lines in both the direction of the *x*-axis and the *y*-axis are zero, or else one of these slopes fails to exist. If we find points that meet this criteria, then those are our candidates for local maximums and minimums.

Let's now go back to the function $z = f(x, y) = -x^2 - y^2 + 4$ and rethink how we might find tangent lines at a point. Previously, back in chapter 3, we went through a somewhat complicated procedure. That is, we picked a point, we fixed either the *x*-

or *y*-value, we graphed the resulting equation in two dimensions, we took a derivative to find the slope of the tangent line at a particular point, and then we moved the whole thing back into three dimensions. Trust me, there's an easer way to do it, and we're all about life being easy! Instead of fixing a *y*-value and then differentiating with respect to *x*, let's just pretend that *y* is fixed and go ahead and find our derivative. Thus, if $z = f(x, y) = -x^2 - y^2 + 4$, but we pretend that *y* is a fixed constant, then when we differentiate, we get back $-2x$. In other words, if *y* is a constant, then the derivative of $-y^2$ is zero just as the derivative of the constant term 4 is zero. Now let's go back and pretend this time that it's *x* which is fixed to a constant value and *y* that is variable. The derivative of *z* this time will be $-2y$.

Already this may be getting a little confusing if we try to do too much of it in our head, so let's try and introduce some notation in order to simplify things. First, since *z* is a function of two variables, that means there are two derivatives we could compute here, one with respect to *x* and one with respect to *y*. Since each derivative by itself tells only part of the story, we call each derivative a partial derivative, and we have two basic notations that we'll use for partial derivatives. For the partial derivative of $z = f(x, y)$ with respect to *x*, we'll use z_x or f_x or $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$. Similarly, for the partial derivative of $z = f(x, y)$ with respect to *y*, we'll use z_y or f_y or $\frac{\partial z}{\partial y}$ or

$\frac{\partial f}{\partial y}$. The latter notation is similar to the usual notation for a derivative except that we use a somewhat stylized version of the letter "d" to denote that it's a partial derivative. If you use this notation, write it correctly so that people will know that it is a partial derivative that you are talking about! Also, both of these partial derivatives are technically defined in terms of limits such as the ones below.

$$z_x = \frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$z_y = \frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

In practice, though, you find the partial derivative with respect to x by treating y as fixed, and you find the partial derivative with respect to y by treating x as fixed. Also, the partial derivative of z with respect to x can be interpreted as either the instantaneous rate of change of z with respect to a change in x, or as the slope of the tangent line in the direction of the x-axis if you evaluate this partial derivate at a particular point. Likewise, the partial derivative of z with respect to y can be interpreted as either the instantaneous rate of change of z with respect to a change in y, or as the slope of the tangent line in the direction of the 0-axis. Since it's important to be very good at computing partial derivatives, below are several examples. Study them well, and make sure you understand what is going on.

1. $z = f(x,y) = x^2 + y^2$

$$\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial z}{\partial y} = 2y$$

2. $z = f(x,y) = x^2 - y^2$

$$\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial z}{\partial y} = -2y$$

3. $z = f(x,y) = xy$

$$\frac{\partial z}{\partial x} = y$$

$$\frac{\partial z}{\partial y} = x$$

4. $z = f(x,y) = x + y$

$$\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial y} = 1$$

5. $z = f(x,y) = \dfrac{y}{x}$

$$\frac{\partial z}{\partial x} = \frac{-y}{x^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x}$$

6. $z = f(x,y) = \ln(xy)$

$$\frac{\partial z}{\partial x} = \frac{y}{xy} = \frac{1}{x}$$

$$\frac{\partial z}{\partial y} = \frac{x}{xy} = \frac{1}{y}$$

7. $z = f(x,y) = e^{5xy^2}$

$$\frac{\partial z}{\partial x} = e^{5xy^2} \cdot 5y^2 = 5y^2 e^{5xy^2}$$
$$\frac{\partial z}{\partial y} = e^{5xy^2} \cdot 10xy = 10xye^{5xy^2}$$

8. $z = f(x,y) = 4x^2 + y^2$

$$\frac{\partial z}{\partial x} = 8x$$
$$\frac{\partial z}{\partial y} = 2y$$

9. $z = f(x,y) = -\left(x^2 + y^2\right)$

$$\frac{\partial z}{\partial x} = -2x$$
$$\frac{\partial z}{\partial y} = -2y$$

10. $z = f(x,y) = x^2$

$$\frac{\partial z}{\partial x} = 2x$$
$$\frac{\partial z}{\partial y} = 0$$

11. $z = f(x,y) = \sqrt{xy}$

$$\frac{\partial z}{\partial x} = \frac{1}{2}(xy)^{-\frac{1}{2}} \cdot y = \frac{y}{2\sqrt{xy}}$$
$$\frac{\partial z}{\partial y} = \frac{1}{2}(xy)^{-\frac{1}{2}} \cdot x = \frac{x}{2\sqrt{xy}}$$

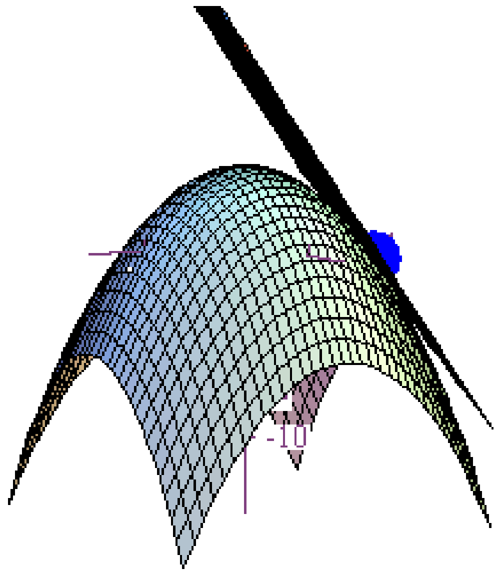
Now let's return to an exploration of $z = f(x,y) = -x^2 - y^2 + 4$ and see how we might construct the tangent plane at the point $(1,2,-1)$. Recall now that back in chapter 1 we said that if an equation for a plane was written in the form $z = Ax + By + C$, then A

would be the slope of the plane in the direction of the *x*-axis, and *B* would be the slope of the plane in the direction of the *y*-axis. We can now find these slopes for $z = f(x, y) = -x^2 - y^2 + 4$ by taking partial derivatives and evaluating the results at the point $(1, 2, -1)$. Clearly, $z_x = -2x$ and $z_y = -2y$. If we evaluate these partial derivatives at $x = 1$ and $y = 2$, we get $z_x(1) = -2(1) = -2$, and $z_y(2) = -2(2) = -4$. Thus, the equation for our tangent plane is starting to look like $z = -2x - 4y + C$. To now find the value of *C*, just plug in the coordinates of the point $(1, 2, -1)$ for *x*, *y*, and *z*.

$$-1 = -2(1) - 4(2) + C \Rightarrow C = 9$$

$$z = -2x - 4y + 9$$

If we now graph our paraboloid, our point, and our surface together, then we can see it worked.

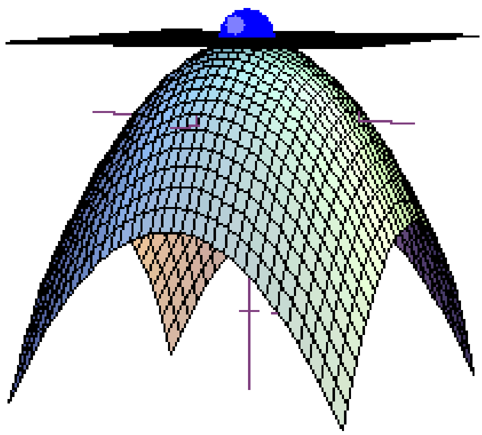


Now let's backtrack a second and think in terms of how we would use our partial derivatives to find the location of our local maximum. As we alluded to previously, at such an extreme point either both partial derivatives equal zero of one of them fails to exist. Well, if we set each of the partial derivatives for $z = f(x, y) = -x^2 - y^2 + 4$ equal to zero, then we're in business.

$$\frac{\partial z}{\partial x} = -2x = 0 \Rightarrow x = 0$$

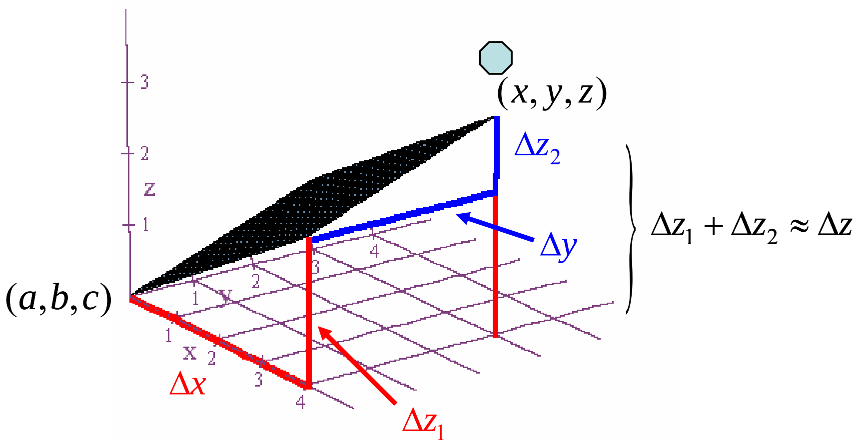
$$\frac{\partial z}{\partial y} = -2y = 0 \Rightarrow y = 0$$

Both partials are equal to zero when $x = 0$ and $y = 0$, so we'll call the point $(0,0)$ a <u>critical point</u>, and from the graph above we can see that we will have a local (and absolute!) maximum at this point. Also, technically speaking, the local maximum is just the *z*-value that we get at our critical point, in this case $z = f(0,0) = 4$. However, I always like to know not only the *z*-coordinate, but the *x*- and *y*-coordinates, too. Thus, I'll always give my answers as coordinates for a point, and I'll refer, in this example, to the point $(0,0,4)$ as a local maximum point. And finally, when we add the tangent plane $z = 4$ to our graph, we see that it is perfectly horizontal at the point $(0,0,4)$.



Later on, we'll learn an algebraic test for determining whether a critical point results in a local maximum or a local minimum or something else, but for now we'll combine our algebraic methods with graphical insights.

At this point, we want to move on to another topic and define what's known as the total differential. It is basically a formula that shows the relationship between small changes in x and y and the corresponding change in z, and among other things, it can be used for doing approximations related to the output of our function. We'll begin our development by looking at a diagram that's similar to some we've seen before.



What we are looking at here is a portion of a tangent plane that we might have at a point (a,b,c) on the surface of some function $z = f(x,y)$. The graph of $z = f(x,y)$, however, is not depicted here. All we are looking at is the tangent plane. Also, assume that (x,y,z) is another point on the graph of $z = f(x,y)$ that is close to (a,b,c). Then the change in z is $\Delta z = z - c$, and using the tangent plane we can approximate this change in z by $\Delta z \approx \Delta z_1 + \Delta z_2$. Also, from our diagram above, we can see that the slope of the plane in the direction of the positive x-axis is $\frac{\Delta z_1}{\Delta x}$ and the slope of the plane in the direction of the positive *y*-axis is $\frac{\Delta z_2}{\Delta y}$. However, these

slopes are also equal, respectively, to $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for our function $z = f(x, y)$. Thus,

$$\frac{\Delta z_1}{\Delta x} = \frac{\partial z}{\partial x} \Rightarrow \Delta z_1 = \frac{\partial z}{\partial x}\Delta x$$

$$\frac{\Delta z_2}{\Delta y} = \frac{\partial z_2}{\partial y} \Rightarrow \Delta z_2 = \frac{\partial z}{\partial y}\Delta y$$

And the consequence of this is that,

$$z - c = \Delta z \approx \Delta z_1 + \Delta z_2 = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y .$$

If we write these changes in *x*, *y*, and *z* as differentials, then we get an expression that we call the <u>total differential</u>.

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

However, since we are mainly going to use this expression for approximations, the previous form, $z - c = \Delta z \approx \Delta z_1 + \Delta z_2 = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$, is a little better. For example, if we take $z = f(x, y) = -x^2 - y^2 + 4$ and $x = 1$ and $y = 2$, and if we want to change our input to $x = 1.1$ and $y = 2.3$ then we can use the total differential to approximate both the new function value and the change that occurs in *z*. For the change in *z* use the formula $\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$. To finish the computation, we have to specify the change in *x*, the change in *y*, and the value of both partial derivatives at our first point, $(1,2)$.

The results are below.

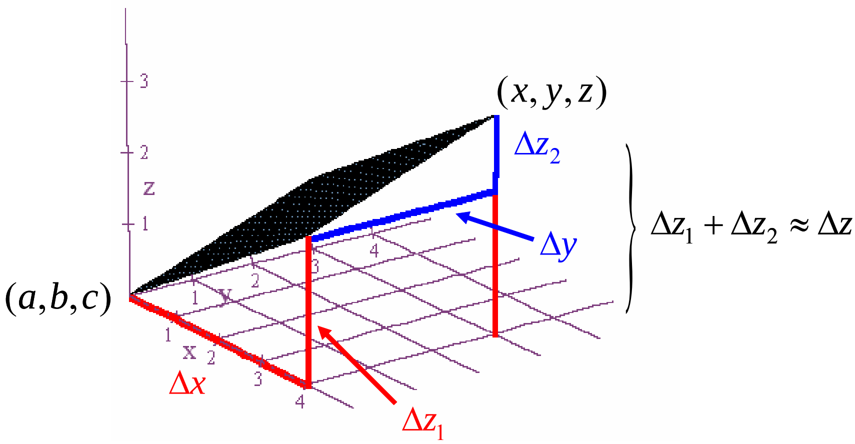
$$\begin{aligned}
&\Delta x = 0.1\\
&\Delta y = 0.3\\
&\frac{\partial z}{\partial x} = -2x,\ \frac{\partial z(1,2)}{\partial x} = -2\\
&\frac{\partial z}{\partial y} = -2y,\ \frac{\partial z(1,2)}{\partial y} = -4\\
&\Delta z \approx \frac{\partial z(1,2)}{\partial x}\Delta x + \frac{\partial z(1,2)}{\partial y}\Delta y = (-2)(0.1) + (-4)(0.3) = -1.4
\end{aligned}$$

We can also approximate the new *z* value by rewriting our formula above as $z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + c$. In this case, we get $z(1.1,\ 2.3) \approx (-2)(0.1) + (-4)(0.3) - 1 = -2.4$. If we computer the actually value of $z(1.1,\ 2.3)$, we get $z(1.1,\ 2.3) = -(1.1^2) - (2.3^2) + 4 = -2.5$ which means the actual change in *z* is -1.5. Well, we can see that our approximations are pretty close, and that's the whole point. Nonetheless, the total differential has some other applications, too. It helps us see at a glance what the chain rule should look like for a function of several variables.

Recall that in first semester calculus you occasionally had a function that could be considered a composition of two or more functions, and then you had to use the chain rule in order to get the derivative. Likewise, for functions of two or more variables there also exists a version of the chain rule, several in fact. Thus, it'll be lots of fun! From our derivation of the total differential, you can pretty much guess the correct form for the chain rule. For example, suppose that *z* is a function of two variables and that *x* and *y* are both functions of one variable, *t*. In other words, $x = x(t)$, $y = y(t)$, and $z = f(x,y) = f\left(x(t), y(t)\right)$, and we want to find $\frac{dz}{dt}$. Well, since the total differential is $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$, a good (and correct!) guess for $\frac{dz}{dt}$ would be

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

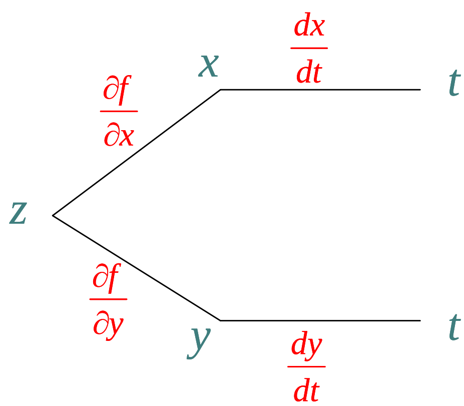
Let's go back to my favorite diagram and look at this in a little more detail.



Recall that from this diagram we derived the formula $\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$. Now just divide everything by Δt to get $\frac{\Delta z}{\Delta t} \approx \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t}$. And finally, take limits as $\Delta t \to 0$ and you get,

$$\frac{dz}{dt} = \lim_{\Delta t \to 0}\frac{\Delta z}{\Delta t} = \lim_{\Delta t \to 0}\left(\frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t}\right) = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

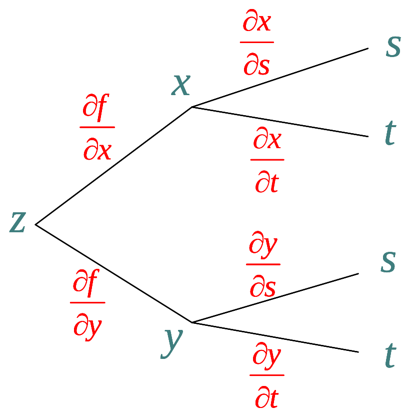
And that's our chain rule! There are a couple of additional points you need to be aware of, though. First, when do we use partial derivative notation and when do we use regular derivative notation? Well, the rule is that if a variable is a function of more than one input, then you use partial derivative notation, and if it's a function of a single input, then you use regular derivative notation. Up above, at first glance we have *z* as a function of two variables, *x* and *y*. However, each of these variables can be written as a function of just one variable, *t*, and so ultimately *z* is a function of a single input *t*. That's why we write $\frac{dz}{dt}$ as an ordinary derivative. We can also draw diagrams such as the one below to help us.



$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

To get the appropriate chain rule, we multiply along the branches and then add together all the results.

If we have different functions, however, then we might need to use different versions of the chain rule. For example, suppose we have $z = f(x, y)$, $x = x(s,t)$, and $y = y(s,t)$ and that we want to find the derivative of z with respect to t. Then this derivative will be a partial derivative since we won't be able to express z solely as a function of t. Our tree diagram for the chain rule in this case is as follows.



$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Now let's suppose that $z = x^2 + y^2$, $x = t^2$, and $y = t^3$, and that we want to find $\frac{dz}{dt}$.

Then by the chain rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2x)(2t) + (2y)(3t^2) = (2t^2)(2t) + (2t^3)(3t^2) = 4t^3 + 6t^5$$

A question you should be asking yourself now is, "Couldn't we have also done this just by first writing *x* and *y* in terms of *t* and then differentiating without having to use the chain rule?" The answer to this question is "yes" as you can see below.

$$z = x^2 + y^2 = (t^2)^2 + (t^3)^2 = t^4 + t^6$$

$$\frac{dz}{dt} = 4t^3 + 6t^5$$

However, the chain rule is also going to be important to us for the role that it plays in helping us prove some key theorems and to derive other important results. For example, suppose we have the equation $xy^3 + x^2 + x^3y = 0$, and suppose that *y* is a function of *x*. Then we can use the chain rule to help us implicitly find the derivative of *y* with respect to *x*. First, if we think of the left-hand side of this equation as defining a function $f(x,y) = xy^3 + x^2 + x^3y$, then the chain rule tells us that

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}.$$

However, since the right-hand side of our original equation is equal to zero, we can set $\frac{\partial f}{\partial x} = 0$ to get,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0.$$

From here it's a simple task to solve for $\frac{dy}{dx}$ to get,

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

See how easy the chain rule made that derivation?

Another easy fact to now derive from the chain rule is the product rule for derivatives that everyone learns back in calculus of a single variable. The only thing we'll do different is that instead of thinking of $y = y(x) = f(x) \cdot g(x)$ as a function of one variable, we'll now think of it as a function of two, *f* and *g*. In other words, $y = y(f,g) = f \cdot g = f(x) \cdot g(x)$. A quick application of the chain rule for functions of several variables will immediately yield from this the familiar product rule.

$$\frac{dy}{dx} = \frac{\partial y}{\partial f}\frac{df}{dx} + \frac{\partial y}{\partial g}\frac{dg}{dx} = g\frac{df}{dx} + f\frac{dg}{dx} = f\frac{dg}{dx} + g\frac{df}{dx}.$$

Now we want to look at another useful tool that is known as the gradient. For a function $z = f(x,y)$ we define the gradient as $grad\ f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$. Notice that the gradient is a vector that is formed using our first partial derivatives. Also, the notation $"\nabla f"$ is generally read as "del f." Furthermore, if we have a function of three variables such as $w = f(x,y,z)$, then we would have $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{j}$. One of the important applications of the gradient vector is the following theorem.

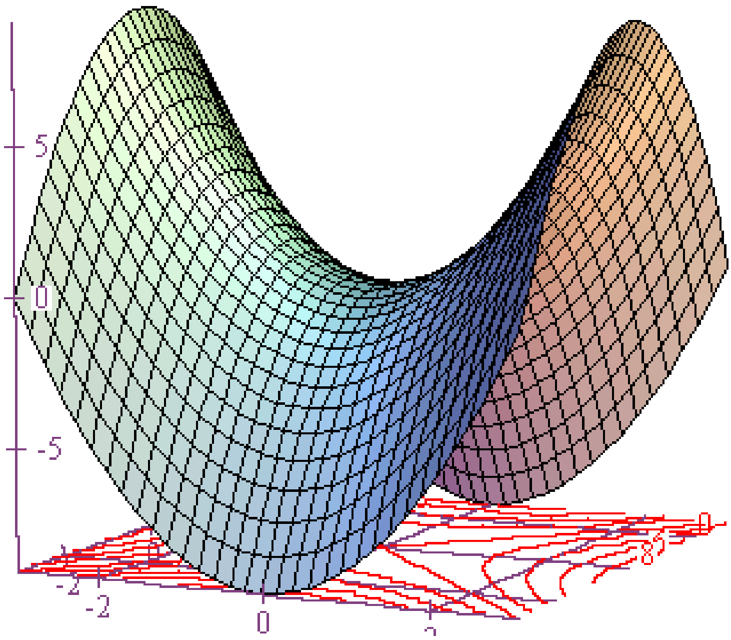
Theorem: Let $z = f(x,y)$ be differentiable at (a,b), and suppose $f(a,b) = c$. Also, let *C* be the level curve $f(x,y) = c$ that passes through (a,b) in the *xy*-plane. If *C* is smooth with a smooth parametrizatoin $\vec{r}(t)$ and if $\nabla f(a,b) \neq 0$, then $\nabla f(a,b)$ is normal to *C* at (a,b). In other words, ∇f is perpendicular to $\vec{r}(t)$ at (a,b).

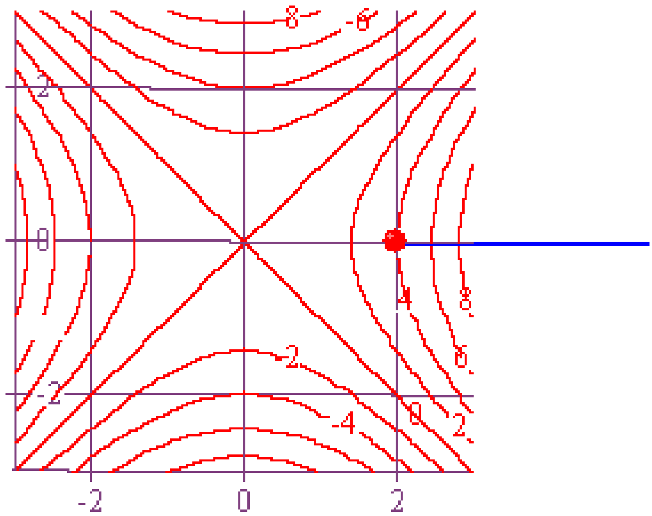
Proof: Let $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ with $t \in I$, an interval, be a smooth parametrization for *C*. Then $f(x,y) = f\left(x(t), y(t)\right) = c$ when $t \in I$. Hence,

$$0 = \frac{dc}{dt} = \frac{df\left(x(t), y(t)\right)}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \nabla f \bullet \frac{d\vec{r}}{dt}$$

Therefore, ∇f is normal to *C* at (a,b). ■

Notice how we used the chain rule in the above proof. That's one of the reasons it's so important. Let's now do a construction to make this all a little more concrete. For our function, let's use $z = x^2 - y^2$, the level curve $z = 4$, and for input we'll use the point $P = (2,0)$. Then $\nabla z = \frac{\partial z}{\partial x}\hat{i} + \frac{\partial z}{\partial y}\hat{j} = 2x\hat{i} - 2y\hat{j}$, and $\nabla z(2,0) = 4\hat{i}$. To plot this vector as a displacement vector in the *xy*-plane starting at $(2,0)$, we can use the following parametric equations: $x = 2 + 4t,\ y = 0,\ 0 \leq t \leq 1$. And below we have the graph of our surface as well as the gradient vector at the point $(2,0)$ on the level curve corresponding to $z = 4$.


5
0
-5
-2
-2
0


-8
-6
2
0
-2
-2
0
2
4
8
-2
6
-4
0
2

Another success! It sure looks to me like the gradient is perpendicular to our level curve at the specified point.

Now here's something which is both a little different and very important. Let's do some constructions this time with the function $z = f(x, y) = -x^2 - y^2 + 4$ at the point $(1, 2, -1)$. We looked at this function earlier in this chapter and found that the equation for the tangent plane at the point $(1, 2, -1)$ was $z = -2x - 4y + 9$ where $-2 = \dfrac{\partial f(1,2)}{\partial x}$ and $-4 = \dfrac{\partial f(1,2)}{\partial y}$, the slopes of tangent lines of $z = f(x, y) = -x^2 - y^2 + 4$, respectively, in the directions of the positive *x*- and *y*-axis when evaluated at the point $(1, 2, -1)$. If we now rewrite our equation as $0 = -2x - 4y - z + 9$, then from past discussions we know that the coefficients of *x*, *y*, and *z* give us a vector that is normal to this plane. In other words, the vector $\vec{v} = -2\hat{i} - 4\hat{j} - \hat{k}$ is perpendicular to the tangent plane to our surface $z = f(x, y) = -x^2 - y^2 + 4$ at the point $(1, 2, -1)$. But notice this, if we set $w = -x^2 - y^2 - z + 4$, then the surface $0 = -x^2 - y^2 - z + 4$ is just a level surface for this function of three variables. Furthermore, the gradient of *w* is $\nabla w = -2x\hat{i} - 2y\hat{j} - \hat{k}$, and evaluated at the point $(1, 2, -1)$ we get $\nabla w(1, 2, -1) = -2\hat{i} - 4\hat{j} - \hat{k}$. But this is the same vector $\vec{v} = -2\hat{i} - 4\hat{j} - \hat{k}$ that is perpendicular to our tangent plane! In other words, we have the following general result.

<u>Theorem:</u> If $w=c$ is a level surface for the function $w=f(x,y,z)$, then the gradient of *w*, ∇w, evaluated at a point *P* on this level surface is perpendicular to the tangent plane at *P* on that surface.

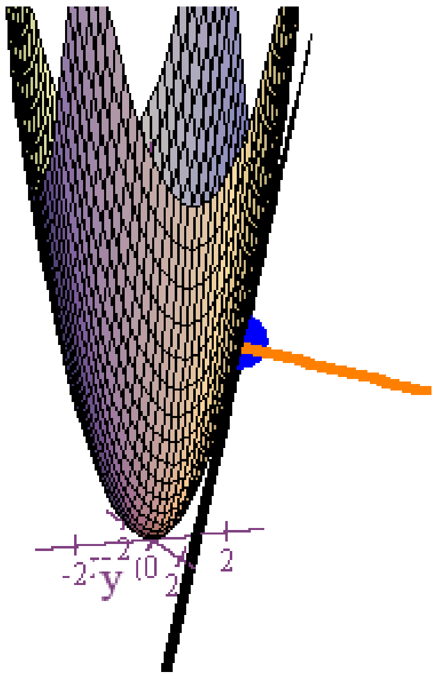
Let's do another construction as an example because this gives us yet another way to find a tangent plane. We can first find a gradient vector that is normal to the tangent plane, and then use the dot product to find the equation for the tangent plane. Here's how. Suppose $z=x^2+y^2$ and $P=(1,2,5)$. Then *P* is a point on the surface of *z*. Notice, also, that we can write the equation for our surface as $0=x^2+y^2-z$. Now let $w=x^2+y^2-z$. Then we can think of $0=x^2+y^2-z$ as just a level surface for the function $w=x^2+y^2-z$. Furthermore, $\nabla w=2x\hat{i}+2y\hat{j}-\hat{k}$, and hence, $\nabla w(1,2,5)=2\hat{i}+4\hat{j}-\hat{k}$ is normal to our surface at the point *P*. Now consider the tangent plane to our surface at the point $P=(1,2,5)$, and suppose $Q=(x,y,z)$ is another point in that plane. Then since $\nabla w(1,2,5)=2\hat{i}+4\hat{j}-\hat{k}$ is perpendicular to the plane, it is also perpendicular to the displacement vector $\overrightarrow{PQ}=(x-1)\hat{i}+(y-2)\hat{j}+(z-5)\hat{k}$ Therefore, $\nabla w(1,2,5)\bullet\overrightarrow{PQ}=0$. But this gives us

$$\begin{aligned}0&=\left(2\hat{i}+4\hat{j}-\hat{k}\right)\bullet\left((x-1)\hat{i}+(y-2)\hat{j}+(z-5)\hat{k}\right)=2(x-1)+4(y-2)-(z-5)\\&=2x+4y-z-5\end{aligned}$$

In other words, $z=2x+4y-5$ is the tangent plane to our surface at $P=(1,2,5)$.

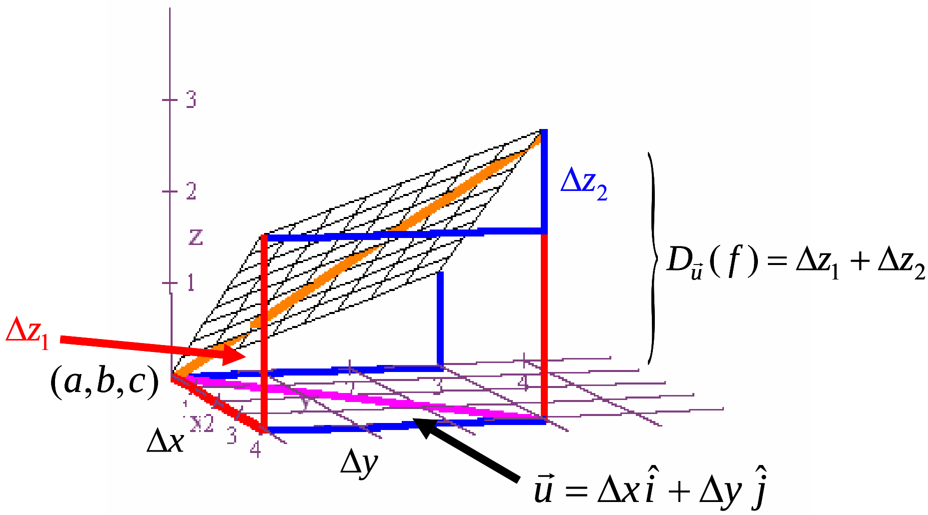
Now let's graph everything. Our surface is $z = x^2 + y^2$, our tangent plane is $z = 2x + 4y - 5$, and the parametric equations for graphing our gradient vector at the point $P = (1,2,5)$ are,

$$\begin{aligned} &x = 1 + 2t \\ &y = 2 + 4t \\ &z = 5 - t \\ &0 \le t \le 1 \end{aligned}$$



And there we have it! Level surface, tangent plane, point, and gradient vector that is perpendicular to the level surface.

So far we've looked only at the slopes of tangent lines in the directions of the positive *x*-axis and the positive *y*-axis. These values correspond to $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. However, suppose we want to find the slope of a tangent line in some other direction. If we do, then we'll refer to this slope as a directional derivative. Furthermore, if a tangent plane contains all possible tangent lines to a surface at a point, and if the tangent plane can be determined from the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at that point, then we should be able to find the value of a directional derivative directly from the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. And that's exactly what's going to happen. Let's start by looking at yet another variation of my favorite diagram.



In this diagram, we are looking at a tangent plane to some surface at the point (a,b,c), and the orange line represents a tangent line in the direction of a unit vector $\vec{u} = \Delta x\hat{i} + \Delta y\hat{j}$ that is shown in magenta above. Since the length of $\vec{u} = \Delta x\hat{i} + \Delta y\hat{j}$ is 1, it follows that the slope of our tangent line is given just by the height of the right triangle depicted with $\vec{u} = \Delta x\hat{i} + \Delta y\hat{j}$ as the base. In other words, by $\Delta z_1 + \Delta z_2$. But as we've seen previously with this diagram, $\Delta z_1 = \frac{\partial z(a,b)}{\partial x}\Delta x$ and $\Delta z_2 = \frac{\partial z(a,b)}{\partial y}\Delta y$.

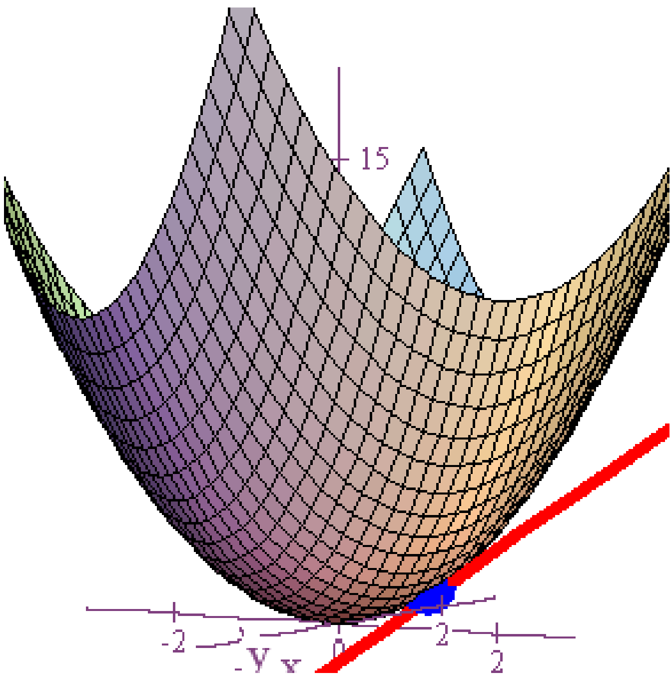
Therefore, the value of the directional derivative of $z = f(x,y)$ in the direction of $\vec{u}$ is

$$D_{\vec{u}}(f) = \Delta z_1 + \Delta z_2 = \frac{\partial f(a,b)}{\partial x}\Delta x + \frac{\partial f(a,b)}{\partial y}\Delta y = \nabla f(a,b) \bullet \vec{u}$$

We usually abbreviate this formula as $D_{\vec{u}}(f) = \nabla f \bullet \vec{u}$. For example, if $z = f(x,y) = x^2 + y^2$ and $\vec{u} = \frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\hat{j}$, then $D_{\vec{u}}(f) = \nabla f \bullet \vec{u} = x\sqrt{2} + y\sqrt{2}$. If we now evaluate this derivative at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, the we get $D_{\vec{u}} f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \nabla f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \bullet \vec{u} = 1 + 1 = 2$. Thus, at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right)$ on the surface of $z = f(x,y) = x^2 + y^2$, the tangent line in the direction of $\vec{u} = \frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\hat{j}$ has slope 2. If we now want to add the graph of this tangent line to our surface graph, it's not too difficult if you think in terms of vectors. Our point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right)$ on the surface corresponds to the position vector $\vec{w} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \hat{k}$. Since our tangent line has slope 2, if we add our unit vector $\vec{u} = \frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\hat{j}$ to this position vector and then go up 2 units by adding on the vector $\vec{v} = 2\hat{k}$, then everything should terminate at another point on the tangent line. In other words, the vector $\vec{u}+\vec{v}$ is parallel to our tangent line. Hence, we can describe the tangent line parametrically as

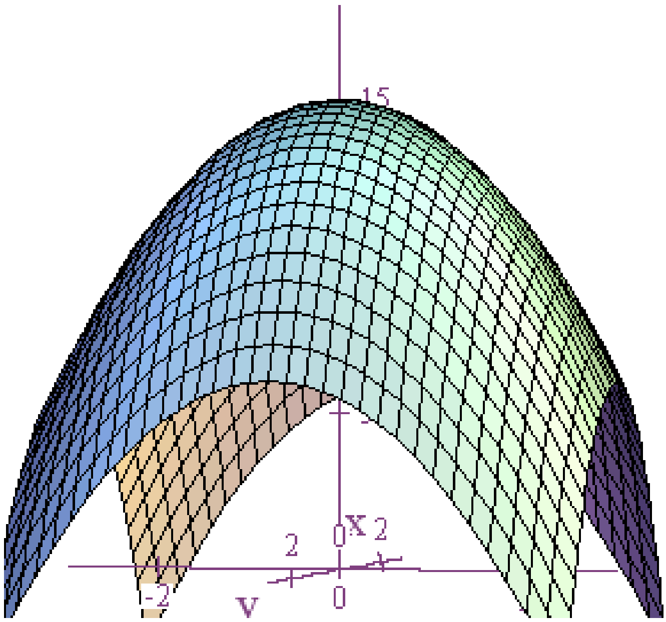
$\vec{w}+t\left(\vec{u}+\vec{v}\right)=\left(\frac{1}{\sqrt{2}}i+\frac{1}{\sqrt{2}}\hat{j}+\hat{k}\right)+t\left(\frac{\sqrt{2}}{2}\hat{i}+\frac{\sqrt{2}}{2}\hat{j}+2\hat{k}\right)$. This, in turn, gives us the following values for x, y, and z.

$$x=\frac{1}{\sqrt{2}}+\frac{t}{\sqrt{2}}$$
$$y=\frac{1}{\sqrt{2}}+\frac{t}{\sqrt{2}}$$
$$z=1+2t$$
$$-\infty<t<\infty$$

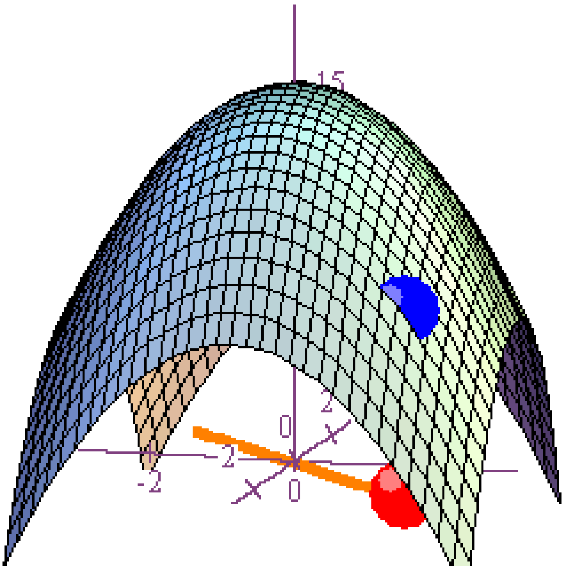


WOW! It actually worked!

If we go back to the formula we derived for computing the directional derivative as a dot product, $D_{\vec{u}}(f) = \nabla f \bullet \vec{u}$, then there are even more wonderful things we can deduce. For instance, using our alternate formula for computing the dot product, we have $D_{\vec{u}}(f) = \nabla f \bullet \vec{u} = \|\nabla f\|\|\vec{u}\|\cos\theta = \|\nabla f\| \cdot 1 \cdot \cos\theta = \|\nabla f\|\cos\theta$. Among other things, this formula tells us that the directional derivative will take on its maximum positive value at a point on the surface when the angle between the gradient vector and the unit vector is zero, and this derivative will take on it's negative value of largest magnitude when the angle between the gradient vector and the unit vector is $180°$ (or π). In other words, if you are trying to climb a hill, then the gradient vector points in the direction of steepest ascent. I must confess that this really confused me when I first heard it mentioned decades ago in multivariable calculus because I thought to myself, "If I want to go up the hill as fast as possible, shouldn't I go up in the direction of 'up'?" Well, yes, that's true. If you want to go up, then go up. However, the type of direction we are speaking of with regard to the gradient vector is more like a compass direction. Consequently, if you are facing north and there is a mountain in front of you, then the direction you go in to quickly ascend the mountain is north, not up. And if you want to descend the mountain as quickly as possible, then you go in the opposite direction, south. As an example, let's look at the mountain $z = -x^2 - y^2 + 15$ below.



The top of this mountain is at $(0,0,15)$, and if we want to ascend the mountain as quickly as possible, then our direction in the plane should always be pointing towards the *z*-axis. Now let's find the gradient vector and evaluate it at the point $(2,2,7)$. We have that $\nabla z = -2x\hat{i} - 2y\hat{j}$ and $\nabla z(2,2,) = -4\hat{i} - 4\hat{j}$. In the graph below, we've plotted a blue dot at $(2,2,7)$ and a red dot right below it at $(2,2,0)$. Furthermore, we've attached our gradient vector to the red dot, and sure enough, it's pointing right toward the *z*-axis. So as we predicted, if we want to ascend this hill as quickly as possible, then our compass direction should always be towards the *z*-axis.



At this point we've talked a lot about derivatives of $z = f(x, y)$ with respect to *x* and with respect to *y*, but if you think back to your first calculus course then you know that we didn't stop there. After all, once we've differentiated a function once, what could be more fun than differentiating it again! For a function of the form $z = f(x, y)$, there are two first partial derivatives we can compute. How many second partial derivatives, however, are possible? A moment's reflection should tell us that there are four second partial derivatives. This, of course, is because if you differentiate with respect to *x* the first time, then you can differentiate with respect to *y* or *x* the second time. Likewise, if you differentiate with respect to *y* the first time, then you can differentiate with respect to either variable the second time. Thus, for $z = f(x, y)$, there are four second partial derivatives that are available.

What notation do we use for the second partial derivatives? Well, just as we had two different notations for first partial derivatives, such as $\frac{\partial z}{\partial x}$ and z_x, we also have two notations for the second partial derivatives. We write the four second partial derivatives either as $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial y \partial x}$, or $\frac{\partial^2 z}{\partial x \partial y}$, or we write them z_{xx}, z_{yy}, z_{xy}, or z_{yx}. The latter two derivatives in each list are called "mixed partials" because we differentiate first with respect to one variable and then the next. However, there is an order difference you have to be aware of. In the notation $\frac{\partial^2 z}{\partial y \partial x}$, we take our derivatives from right to left. In other words, $\frac{\partial^2 z}{\partial y \partial x}$ means that you first differentiate with respect to *x* and then with respect to *y*. This is because the notation $\frac{\partial^2 z}{\partial y \partial x}$ is just shorthand for $\frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial y}$. On the other hand, when we write z_{yx}, we perform the operations in order from left to right. Thus, this one means that we should first differentiate with respect to *y* and then with respect to *x*. In summary, $\frac{\partial^2 z}{\partial y \partial x} = z_{xy}$,

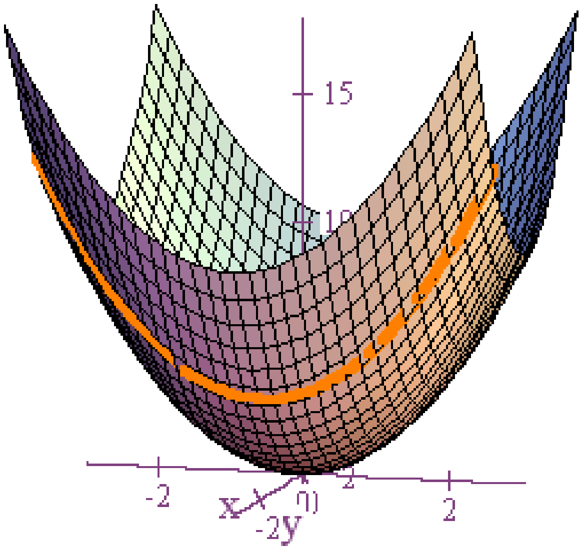
$\frac{\partial^2 z}{\partial x \partial y} = z_{yx}$, $\frac{\partial^2 z}{\partial x^2} = z_{xx}$, and $\frac{\partial^2 z}{\partial y^2} = z_{yy}$.

Let's now take a simple example. If $z = x^2 + y^2$, then $z_x = 2x$ and $z_y = 2y$. If we now compute our second partials, we get $z_{xx} = 2$, $z_{xy} = 0$, $z_{yx} = 0$, and $z_{yy} = 2$. For clarity, we often like to arrange these second partial derivatives in a matrix where they always appear in the following order.

$$\begin{pmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

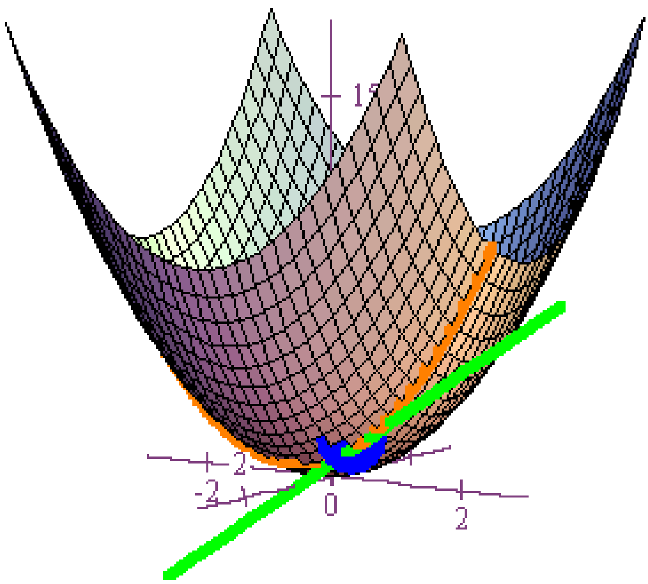
Notice in this example that $z_{xx} = z_{yy}$ and $z_{xy} = z_{yx}$. The first equality is simply by accident, but the second one, $z_{xy} = z_{yx}$, happens almost all the time. In fact, there is a theorem that says that if the mixed partials are continuous at a point in the interior of the domain of our function, then the mixed partials will be equal at that point. And since in calculus, we tend to deal with functions that are continuous almost everywhere, we tend to almost always have equality of the mixed partials. Thus, we can use this property as a check to make sure that we've computed our mixed partials correctly.

How do we interpret second partial derivatives? Again, pretty much the same as an ordinary derivative. Just as in first semester calculus, z_x tells us the rate at which *z* is changing with respect to *x*, and if we take a cross-section by fixing a value for *y*, then for that fixed *y* value, z_x will tell us over what intervals, with respect to *x*, *z* is increasing and where it is decreasing. If we continue on to the second partial z_{xx}, then this will tell us something about the concavity of our cross-section. For example, with $z = x^2 + y^2$, we had $z_x = 2x$. This tells us that regardless of what we set *y* equal to, the *z* values in the corresponding cross-section will be decreasing when *x* is negative, increasing when *x* is positive, and we'll have a horizontal tangent line when $x = 0$. Differentiating with respect to *x* a second time gives us $z_{xx} = 2$. This is now telling us that now matter what value we fix *y* at, the cross-section will be concave-up. Below we see a cross-section corresponding to $y = -2$, and as predicted, the graph of the cross-section is decreasing for $x < 0$ and increasing for $x > 0$, and for all values of *x*, the graph is concave-up.

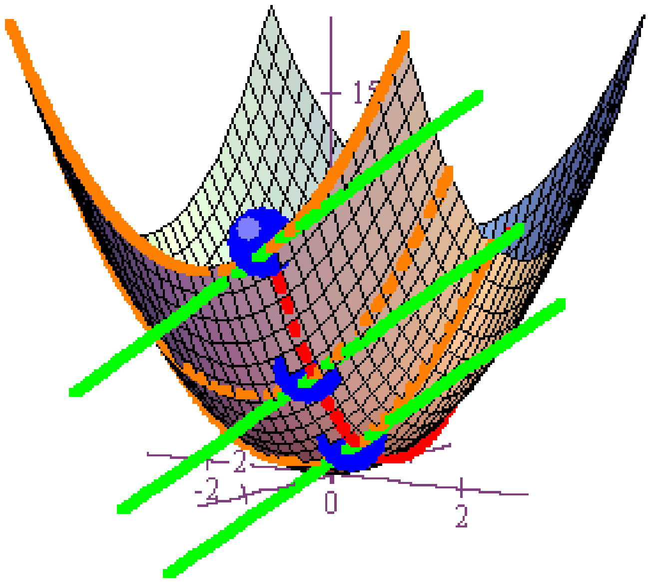


In a similar manner, z_{yy} will tell us something about the concavity of a cross-section obtained by fixing a value of *x*.

Now how do we interpret the mixed partials z_{xy} and z_{yx}. On the one hand, z_{xy} tells us the rate at which $\frac{\partial z}{\partial x}$ changes with respect to a change in *y*, and z_{yx} tells us the rate at which $\frac{\partial z}{\partial y}$ changes with respect to a change in *x*. That part is clear simply from our understanding that derivatives are always instantaneous rates of change. But on the other hand, these mixed partials are much harder to visualize geometrically. In the case of $z = x^2 + y^2$, it's pretty easy. For this function, we have $z_{xy} = 0 = z_{yx}$. This means that if we first find z_x and then look to see how this rate changes as we let y change, we discover that it doesn't change at all. The rate of change is zero. For example, let's take $z = x^2 + y^2$, fix things at a point $P = (1,1,2)$ on the graph, and look at the cross-section corresponding to $x = 1$. Below is the graph of our surface along with the cross-section, and the tangent line at the point $P = (1,1,2)$.

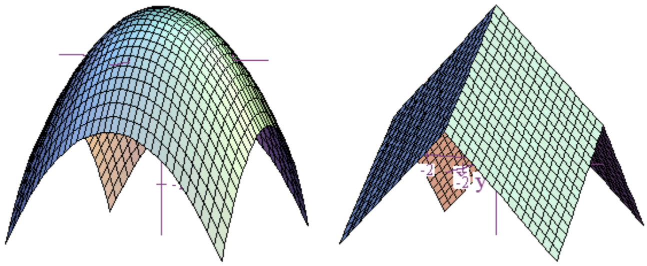


If we now repeat this construction at the fixed values $x = 2$ and $x = 3$, we'll see that the slopes of the tangent lines corresponding to $y = 1$ don't change. And this happens because $z_{yx} = 0$. In other words, the slopes of the tangent lines in the direction of the y-axis don't change as x changes, the rate of change with respect to x is zero. Here's a graph of what we're talking about.

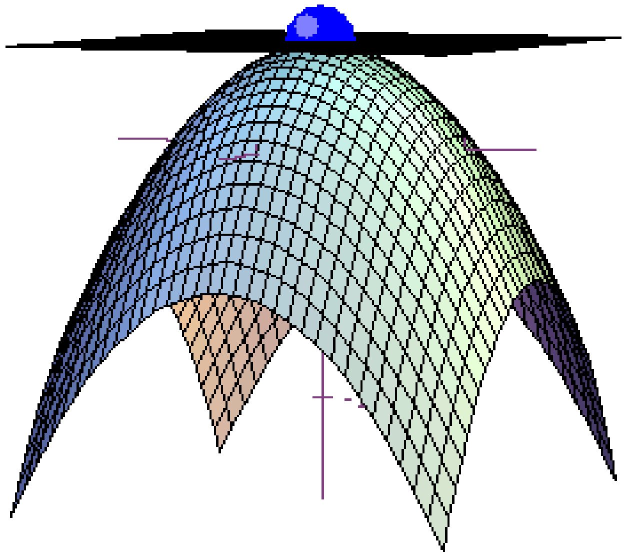


The red curve above corresponds to the cross-section $y = 1$. As we let x vary along this curve, the slopes of the tangent lines in the direction of the y-axis remain unchanged. In other words, their rate of change is zero. This is what $z_{yx} = 0$ is trying to tell us. If this is hard to visualize, don't worry to much about it. It takes practice. Furthermore, the interpretations of z_{xx} and z_{yy} as indicators of concavity will be far more important to us in the long run.

It's finally time for us to discuss the main application of derivatives of functions of several variables. Namely, how do we use these derivatives to find extreme values. Let's revisit two examples we looked at earlier,



The surface on the left is the graph of $z = -\left(x^2 + y^2\right) + 4$, and the surface on the right is the graph of $z = -\left(|x| + |y|\right) + 4$. It's clear that both of these surfaces have a maximum point at $(0,0,4)$. However, it's also clear that we can't define tangent lines at this point on the second surface since we have a sharp corner at that location. Consequently, neither $\frac{\partial z}{\partial x}$ nor $\frac{\partial z}{\partial y}$ exist at that point. On the other hand, both partial derivatives exist at the point $(0,0,4)$ on the graph of $z = -\left(x^2 + y^2\right) + 4$, and both of our first partial derivatives will be zero at this point resulting in horizontal tangent lines that subsequently give rise to a horizontal tangent plane.



What we are seeing here is similar to what you saw in your first calculus course, and we now summarize the results below using the concept of the partial derivative.

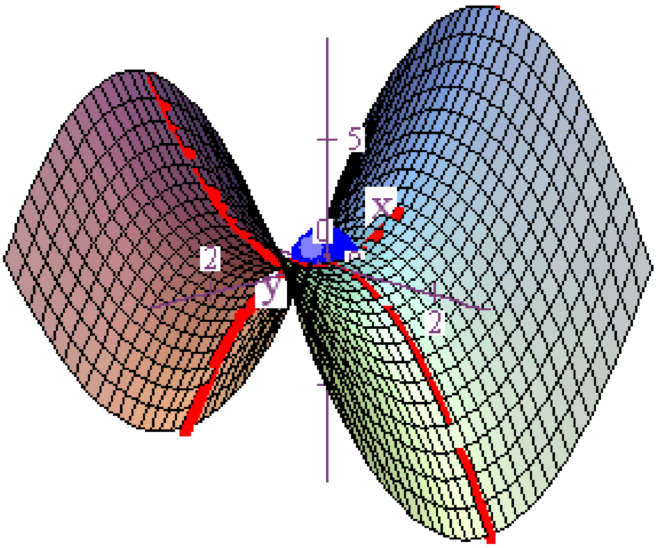
Definition: Let (a,b) be a point contained in an open region R on which a function $z = f(x,y)$ is defined. Then (a,b) is a critical point if either of the following conditions is true:

1. $z_x(a,b) = 0 = z_y(a,b)$
2. $z_x(a,b)$ does not exist
3. $z_y(a,b)$ does not exist.

Theorem: If $z = f(x,y)$ has a relative maximum or a relative minimum at a point (a,b) contained within an open region *R* on which $z = f(x,y)$ is defined, then (a,b) is a critical point.

In other words, critical points are going to be points where either both first derivatives are zero or else one of the first derivatives fails to exist, and if we have a local extreme value, then it has to occur at a critical point. Before we go over a test for helping us classify what type of critical point we have, let's look at a very special kind of point called a saddle point. The classic example of this is found in the graph of $z = x^2 - y^2$. For this function, we have $z_x = 2x$ and $z_y = -2y$. Furthermore,

$\left.\begin{array}{l} z_x = 0 \\ z_y = 0 \end{array}\right\} \Rightarrow \begin{array}{r} 2x = 0 \\ -2y = 0 \end{array} \Rightarrow \begin{array}{l} x = 0 \\ y = 0 \end{array}$. Thus, $(0,0)$ is a critical point. However, if we look at the corresponding point, $(0,0,0)$, on our surface graph, then we easily see that it's neither a maximum nor a minimum point.



Instead, what we see is that this point is at the bottom of one parabolic cross-section and at the top of another parabolic cross-section. When we have a point like this which is a critical point, but movement in one direction causes *z* to increase while movement in another direction causes *z* to decrease, then we call the point on our surface a saddle point. The terminology obviously derives from the saddle shape of the graph above. Thus, when we try to classify the points on our surface corresponding to critical points (a,b), sometimes they will be relative maximum points, sometimes they will be relative minimum points, and sometimes the will be saddle points.

Below is a simple test that will allow us to classify most critical points we come across. The proof of this theorem is rather messy, and it is usually left out of calculus books. However, we'll give a proof at the end of this chapter. Nonetheless, feel free to skip it and move on if you wish.

<u>Second Partials Test:</u> Suppose $z = f(x, y)$ has continuous second partial derivatives on an open region containing a point (a,b) such that $z_x(a,b) = 0 = z_y(a,b)$, and let

$$D = \begin{vmatrix} z_{xx}(a,b) & z_{xy}(a,b) \\ z_{yx}(a,b) & z_{yy}(a,b) \end{vmatrix} = z_{xx}(a,b)z_{yy}(a,b) - z_{xy}(a,b)z_{yx}(a,b).$$

Then:

1. If $D > 0$ and $z_{xx}(a,b) > 0$, $f(a,b)$ is a relative minimum.
2. If $D > 0$ and $z_{xx}(a,b) < 0$, $f(a,b)$ is a relative maximum.
3. If $D < 0$, $(a,b,f(a,b))$ is a saddle point.
4. If $D = 0$, the test is inconclusive.

Let's start with a very simple example, $z = f(x,y) = x^2 + y^2$. If we take the first partial derivatives, we get $z_x = 2x$ and $z_y = 2y$. Setting both of these equal to zero and solving for *x* and *y* yields,

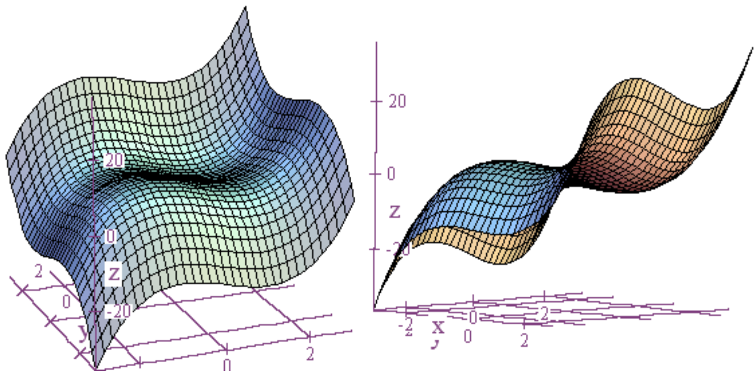
$$\begin{matrix} 2x = 0 \\ 2y = 0 \end{matrix} \Rightarrow \begin{matrix} x = 0 \\ y = 0 \end{matrix}$$

Therefore, $(0,0)$ is our critical point. We now need to find our second partials matrix and evaluate its determinant at the critical point.

$$\begin{pmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow D(0,0) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = (2)(2)-(0)(0) = 4 > 0$$

There are two things to notice here. First, in our second partials matrix we found $z_{xy} = 0 = z_{yx}$. Since we expect to find $z_{xy} = z_{yx}$, we're probably right on track. Additionally, if these two second mixed partials were not equal to each other, then we should suspect that we've made an error. The second important thing to notice is that $D(0,0) = 4 > 0$. This automatically means that we have either a local maximum or a local minimum. To determine which, our second partials test tells us to look at the sign of z_{xx} at our critical point. In this case, we have $z_{xx}(0,0) = 2 > 0$. We can interpret this second derivative as meaning that a particular cross-section of our surface is concave-up at the critical point. Therefore, the critical point is at the bottom of this cross-section, and the point $(0,0,f(0,0)) = (0,0,0)$ is a local minimum point.

Now let's look at $z = f(x,y) = x^3 - 3x + y^3 - 3y$. The graph below suggests that there is one local maximum, one local minimum, and two saddle points.



We'll find all the critical points, but apply our second partials test to just one of them as an example. If we take our first partial derivatives, we get,

$$z_x = 3x^2 - 3 = 3\left(x^2 - 1\right) = 3(x+1)(x-1)$$
$$z_y = 3y^2 - 3 = 3\left(y^2 - 1\right) = 3(y+1)(y-1)$$

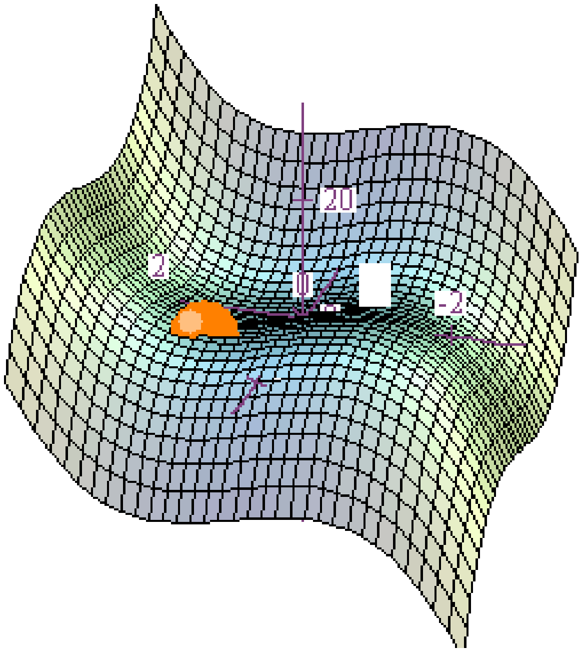
Setting each partial derivative equal to zero results in,

$$\left.\begin{array}{l} z_x = 0 \\ z_y = 0 \end{array}\right\} \Rightarrow \begin{array}{l} 3(x+1)(x-1) = 0 \\ 3(y+1)(y-1) = 0 \end{array} \Rightarrow \begin{array}{l} x = -1, 1 \\ y = -1, 1 \end{array}$$

Therefore, our critical points are $(-1,-1)$, $(-1,1)$, $(1,-1)$, and $(1,1)$. Let's do our second partials test just at the point $(-1,1)$ to see what happens. Setting up our second partials matrix and computing the value of $D(-1,1)$, we get,

$$\begin{pmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{pmatrix} = \begin{pmatrix} 6x & 0 \\ 0 & 6y \end{pmatrix} \Rightarrow D(-1,1) = \begin{vmatrix} -6 & 0 \\ 0 & 6 \end{vmatrix} = (-6)(6) - (0)(0) = -36 < 0$$

And this is as far as we need to go. Since $D(-1,1) < 0$, we automatically know from our second partials test that $\left(-1, 1, f(-1,1)\right) = (-1,1,0)$ is a saddle point. Also, a quick look at this point on our graph confirms this conclusion.



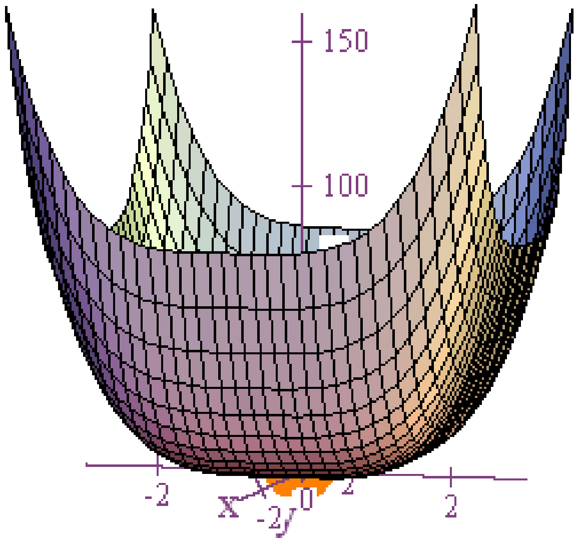
Now let's look at just one more function, $z = f(x,y) = x^4 + y^4$. Taking first partial derivatives, we have $z_x = 4x^3$ and $z_y = 4y^3$. Hence, our critical point is $(0,0)$ since,

$$\left.\begin{array}{l} z_x = 0 \\ z_y = 0 \end{array}\right\} \Rightarrow \begin{array}{l} 4x^3 = 0 \\ 4y^3 = 0 \end{array} \Rightarrow \begin{array}{l} x = 0 \\ y = 0 \end{array}$$

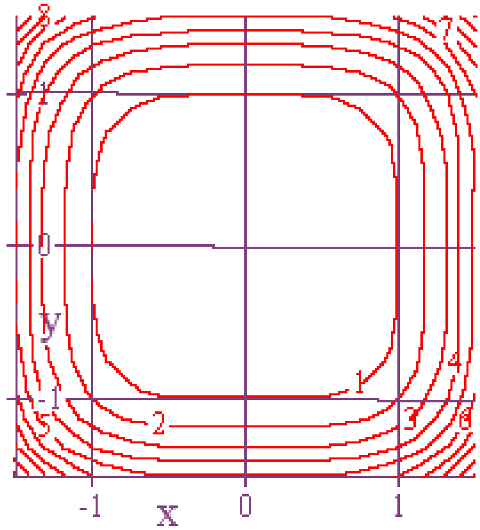
If we now perform the second partials test, we get,

$$\begin{pmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{pmatrix} = \begin{pmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{pmatrix} \Rightarrow D(0,0) = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = (0)(0) - (0)(0) = 0$$

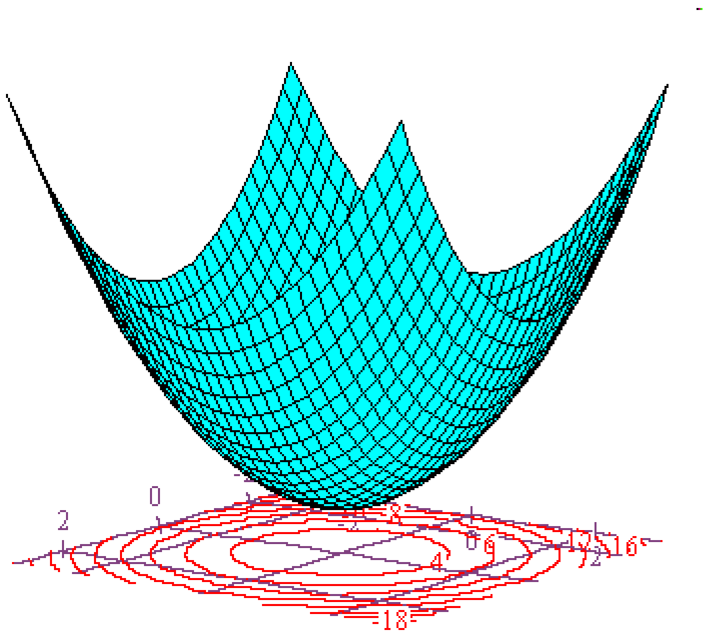
Unfortunately, our second partials test tells us that the test is inconclusive if $D = 0$. However, a glance at the graph will confirm that the point $(0,0,0)$ is a local minimum point.



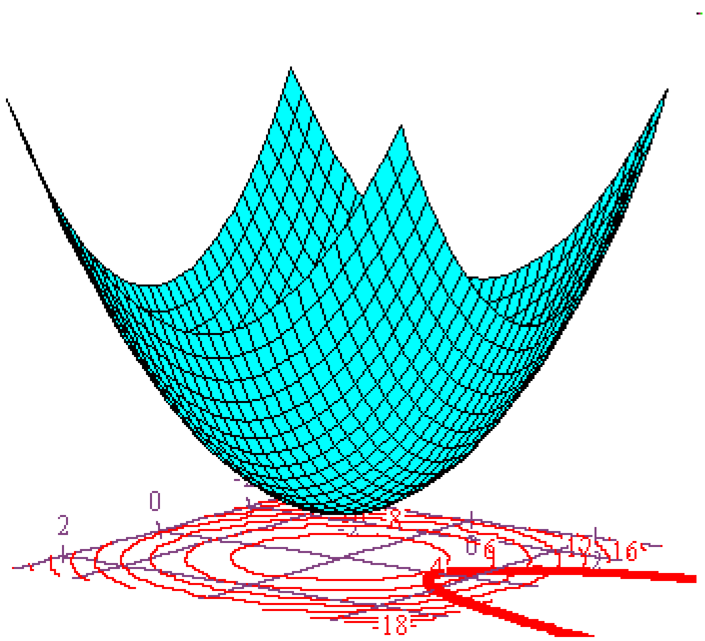
We can also get confirmation by looking at the level curves for this function. Below you can easily see the wagons circling tighter and tighter around the point $(0,0)$, and the *z*-values decrease as we approach the origin.



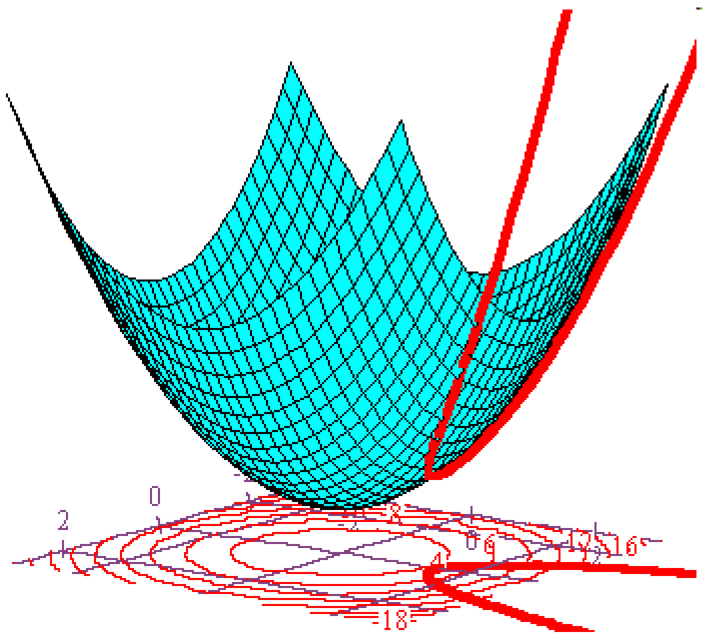
We now want to move on to another type of optimization problem, problems where are input values are restricted to some type of constraint curve. These problems will involve a technique known as *Lagrange multipliers*. To begin, consider the graph of a function $z = f(x, y)$ below .



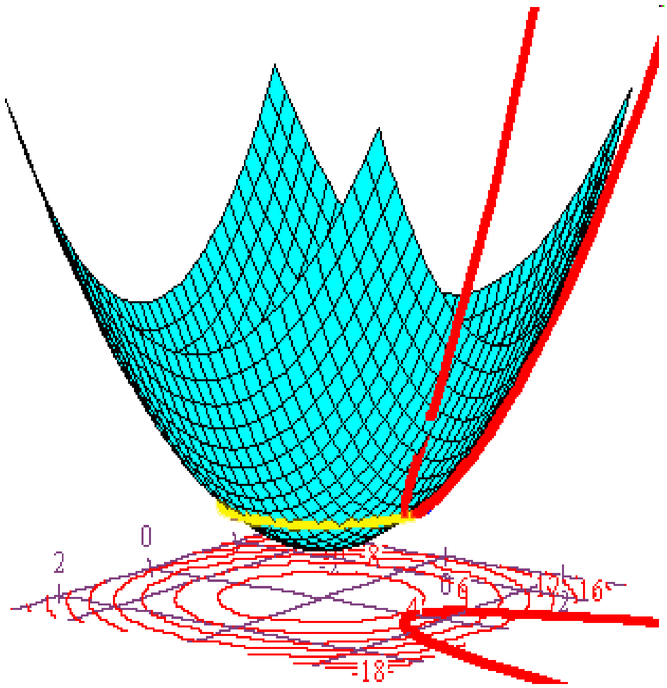
In looking at this graph, it appears that it is defined for all values of *x* and *y*, and that a local (and absolute) minimum will occur at the origin. Now let's add a curve to the *xy*-plane.



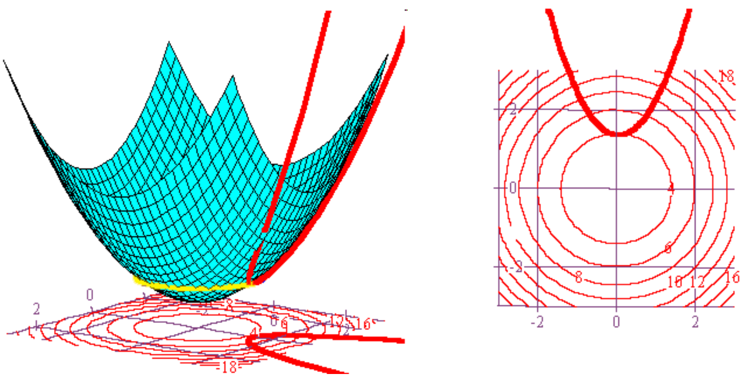
If we restrict our input values just to the points on this curve, then, graphically, the result will be a corresponding curve that lies on our surface.



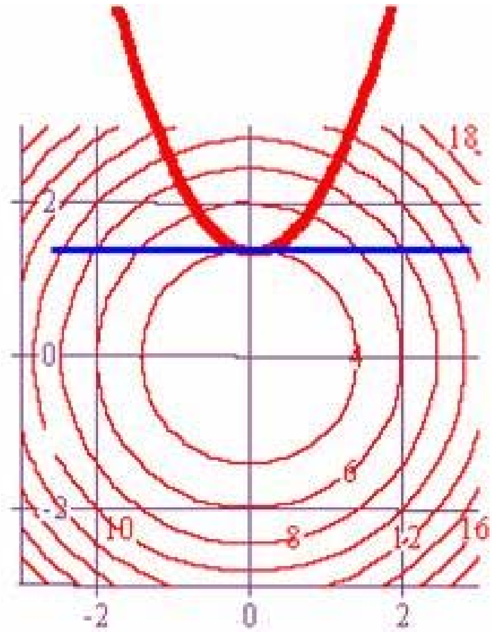
We can see that, with this restricted input, we now have our minimum point residing at a different location. However, what is also (hopefully) obvious is that there is going to be a contour curve on the surface that touches our surface curve right at this minimum point.



If we now move this contour curve down to the *xy*-plane, then it will also touch our constraint curve at a single point.



In other words, the constraint curve in the *xy*-plane and the level curve in the *xy*-plane are tangent to one another, and thus, they have a common tangent line.



Now, what does this all mean to us? It means this. Suppose our we consider our constraint curve as just a level curve for some other function $g(x, y)$. Then as we can see above, there is a level curve for $f(x, y)$ such that $f(x, y)$ and $g(x, y)$ have a common tangent line at the point in the *xy*-plan that corresponds to the minimum point of the curve on our surface graph. However, recall that a gradient vector evaluated at a point is normal to the level curve at that point. Hence, it follows that, at this point of tangency, ∇f is parallel to ∇g. Recall, too, that two vectors are parallel if and only if one is a scalar multiple of the other. If we denote this scalar by the Greek letter lambda, λ, then we get $\nabla f = \lambda \nabla g$. At the partial derivative level we write this as $f_x = \lambda g_x$ and $f_y = \lambda g_y$. This number λ is what is known as a *Lagrange* *multiplier*, named after the great mathematician Joseph Lagrange who discovered this technique. At this point, though, if we add the constraint equation, written as $g(x,y)=c$, to the other two, then we arrive at the following system that we need to solve for *x*, *y*, and λ in order to find our extreme point. In other words, solve the system below, and you're done!

$$\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ g(x,y) &= c \end{aligned}$$

Of course, solving this system is sometimes easier said than done, but let's take a simple example. Suppose $z = f(x,y) = x^2 + y^2$ and our constraint curve is the graph of $y - x^2 = 4$. Then we can think of our constraint curve as a level curve for the function $g(x,y) = y - x^2$. We now have the following equations to play around with.

$$\left.\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ g(x,y) &= c \end{aligned}\right\} \Rightarrow \begin{aligned} 2x &= -\lambda 2x \\ 2y &= \lambda \\ y - x^2 &= 4 \end{aligned}$$

From the first equation we get that either $x = 0$ or $\lambda = -1$. If $x = 0$, then substitution into the last equation tells us that $y = 4$. On the other hand, if $\lambda = -1$, then substitution into the second equation tells us that $y = -\frac{1}{2}$. If we plug this value into the third equation, we get $-\frac{1}{2} - x^2 = 4 \Rightarrow -x^2 = \frac{9}{2} \Rightarrow x^2 = -\frac{9}{2}$. This, however, can't happen since for any real number $x^2 \geq 0$. Hence, our minimum value occurs when

$x=0$ and $y=4$, and the coordinates of our minimum point are $(0,4,16)$. And it's just that simple! Of course, once you understand the method of Lagrange multipliers, the method is simple, but it's solving the actual equations that often times gets very difficult. The problem is that our equations are often nonlinear, and methods that work well on one problem may not help at all on another.

Here's an example that involves a function of three variables. The method of Lagrange multipliers is, nonetheless, the same. We just have to set up an additional equation for the extra variable.

Problem: Use Lagrange multipliers to find the minimum distance between the point $(1,2,3)$ and the plane $4x+5y+6z=20$.

Solution: We'll let the equation for our plane, $4x+5y+6z=20$, be our constraint, and we'll set $z=f(x,y)=(x-1)^2+(y-2)^2+(z-3)^2$. Basically, we're setting $f(x,y)$ equal to the square of the distance between the point $(1,2,3)$ and another point (x,y,z). Certainly, if we can minimize the square of the distance, then we've also found the solution to the minimal distance. However, by working with the square of the distance, the derivative process will be simpler. And finally, we'll set

$g(x, y) = 4x + 5y + 6z$ so that $4x + 5y + 6z = 20$ can be thought of as a level surface for this function. Now let's find some derivatives!

$$f_x = 2(x-1) = 2x - 2$$
$$g_x = 4$$

$$f_y = 2(y-2) = 2y - 4$$
$$g_y = 5$$

$$f_z = 2(z-3) = 2z - 6$$
$$g_z = 6$$

From these derivatives, we apply lagrange multipliers to get the following equations,

$$2x - 2 = 4\lambda$$
$$2y - 4 = 5\lambda$$
$$2z - 6 = 6\lambda$$

If we solve these equations for *x*, *y*, and *z*, then we get,

$$x = \frac{4\lambda + 2}{2}$$
$$y = \frac{5\lambda + 4}{2}$$
$$z = \frac{6\lambda + 6}{2}$$

We can now substitute these expressions for our variables in our constraint equation, and this will allow us to solve for λ.

$$4x+5y+6z=4\left(\frac{4\lambda+2}{2}\right)+5\left(\frac{5\lambda+4}{2}\right)+6\left(\frac{6\lambda+6}{2}\right)=20$$
$$\Rightarrow 16\lambda+8+25\lambda+20+36\lambda+36=40$$
$$\Rightarrow 77\lambda=-24$$
$$\Rightarrow \lambda=-\frac{24}{77}$$

Hence,

$$x=\frac{4\lambda+2}{2}=\frac{59}{77}$$
$$y=\frac{5\lambda+4}{2}=\frac{94}{77}$$
$$z=\frac{6\lambda+6}{2}=\frac{159}{77}$$

Thus, the point $\left(\frac{29}{77},\frac{94}{77},\frac{159}{77}\right)$ should be the point on the plane $4x+5y+6z=20$ that is closest to the point $(1,2,3)$. Let's do a little check, though, before we compute the minimum distance. First to verify that $\left(\frac{29}{77},\frac{94}{77},\frac{159}{77}\right)$ is in the plane, note that $4\left(\frac{29}{77}\right)+5\left(\frac{94}{77}\right)+6\left(\frac{159}{77}\right)=\frac{116}{77}+\frac{470}{77}+\frac{954}{77}=\frac{1540}{77}=20$. Second, it's obvious that if the line segment from $(1,2,3)$ to $\left(\frac{29}{77},\frac{94}{77},\frac{159}{77}\right)$ represents the shortest distance between the point $(1,2,3)$ and the plane $4x+5y+6z=20$, then this line segment should be perpendicular to our plane. Let's do a check of this both algebraically and visually. We already know one vector perpendicular to our plane, the one we obtain from the coefficients of *x*, *y*, and *z*. In other words, $\vec{v} = 4\hat{i} + 5\hat{j} + 6\hat{k}$. Parametric equations for the line that passes through $\left(\frac{29}{77}, \frac{94}{77}, \frac{159}{77}\right)$ in the direction of this vector are,

$$\begin{aligned}
x &= \frac{29}{77} + 4t \\
y &= \frac{94}{77} + 5t \\
z &= \frac{159}{77} + 6t \\
-\infty &< t < \infty
\end{aligned}$$

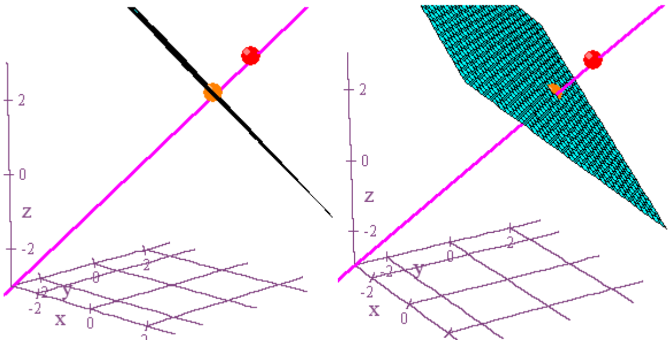
Now let's see what value of *t* will make *x* equal to 1.

$$\frac{29}{77} + 4t = 1 \Rightarrow 4t = 1 - \frac{29}{77} = \frac{48}{77} \Rightarrow t = \frac{12}{77}$$

If we set $t = \frac{12}{77}$ and find the corresponding values for *y* and *z*, then this will show that the point $(1,2,3)$ is on the line that passes through $\left(\frac{29}{77}, \frac{94}{77}, \frac{159}{77}\right)$ and is perpendicular to the plane $4x + 5y + 6z = 20$.

$$x = \frac{29}{77} + 4\left(\frac{12}{77}\right) = \frac{77}{77} = 1$$
$$y = \frac{94}{77} + 5\left(\frac{12}{77}\right) = \frac{154}{77} = 2$$
$$z = \frac{159}{77} + 6\left(\frac{12}{77}\right) = \frac{231}{77} = 3$$

Now, for good measure, let's get some visual confirmation.



Looks confirmed to me! The minimum distance is now

$$d = \sqrt{\left(1 - \frac{29}{77}\right)^2 + \left(2 - \frac{94}{77}\right)^2 + \left(3 - \frac{159}{77}\right)^2} = \sqrt{\frac{2304}{5929} + \frac{3600}{5929} + \frac{5184}{5929}} = \sqrt{\frac{144}{77}}$$
$$= \frac{12}{\sqrt{77}} = \frac{12\sqrt{77}}{77} \approx 1.3675$$

At the start of this section on Lagrange multipliers, we gave a rather visual proof of why it would work for functions of the form $z = f(x, y)$. We would like to give an alternate proof now that is another example of how the chain rule is frequently important in proving theorems in multivariable calculus.

Theorem: Let $f(x,y)$ and $g(x,y)$ have continuous first partial derivatives, and suppose that $f(x,y)$ has an extreme value at the interior point (x_0, y_0) on a smooth constraint curve represented by $g(x,y)=c$. If $\nabla g(x_0,y_0) \neq \vec{0}$, then there is a real number λ such that $\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)$.

Proof: Let $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ be a smooth parametrization for the constraint curve, and suppose $f(x_0,y_0) = f(x(t_0), y(t_0))$ is an extreme value. Then since $f(x,y)$ is differentiable along this curve, $0 = \dfrac{df(x,y)}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} = \nabla f \bullet \dfrac{d\vec{r}}{dt}$ when the derivatives are evaluated at $t = t_0$. Therefore, $\nabla \mathrm{f}(x_0,y_0) \perp \vec{r}\,'(t_0)$. But since $\vec{r}(t)$ is a level curve for $w = g(x,y)$, $\nabla \mathrm{g}\ (x_0,y_0)$ is also perpendicular to $\vec{r}\,'(t_0)$. Therefore, $\nabla \mathrm{f}(x_0,y_0) \,||\, \nabla g(x_0,y_0) \Rightarrow \nabla \mathrm{f}(x_0,y_0) = \lambda \nabla g(x_0,y_0)$. ■

So, that's it for now for Lagrange multipliers, and we're just about through with derivatives. There's just one more thing left to look at, the dreaded proof of the second partials test, so here goes. Gird your loins, and get ready for the ride!

Theorem (Second Partials Test): Suppose (a,b) is a point such that

$f_x(a,b) = 0 = f_y(a,b)$, and let

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b) \right]^2 = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix}.$$

1. If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.
2. If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.
3. If $D < 0$, then $(a,b,f(a,b))$ is a saddle point.
4. If $D = 0$, then we know nothing.

Proof: Let $D(a,b) = f_{xx}(a,b) \cdot f_{yy}(a,b) - \left[f_{xy}(a,b) \right]^2$. Notice that, for the sake of simplicity, in the statement and execution of this proof we are assuming the equality of the two mixed partials, $f_{xy} = f_{yx}$. Also, we will abbreviate the above equality as $D = f_{xx} f_{yy} - f_{xy}{}^2$. Additionally, we will assume that every expression we write down is being evaluated at the point (a,b). Since all directional derivatives at this point will have the value zero, our strategy will be to look at concavity as determined by the sign of the second derivative in order to decide whether $(a,b,f(a,b))$ is a maximum point, a minimum point, or a saddle point.

Now let $\vec{u} = h\hat{i} + k\hat{j}$ be a unit vector. Then $D_{\vec{u}} f = \nabla f \bullet \vec{u} = f_x h + f_y k$. Also,

$$D^2_{\vec{u}} f = D_{\vec{u}}\left(D_{\vec{u}} f\right) = \nabla\left(D_{\vec{u}} f\right) \bullet \vec{u} = \left[\left(f_{xx}h + f_{yx}k\right)\hat{i} + \left(f_{xy}h + f_{yy}k\right)\hat{j}\right] \bullet \left[h\hat{i} + k\hat{j}\right]$$

$= f_{xx}h^2 + f_{yx}hk + f_{xy}hk + f_{yy}k^2 = f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2$. We can rewrite this last expression by completing the square.

$$\begin{aligned}
D^2_{\vec{u}} f &= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 = f_{xx}\left(h^2 + \frac{2f_{xy}}{f_{xx}}hk\right) + f_{yy}k^2 \\
&= f_{xx}\left(h^2 + \frac{2f_{xy}}{f_{xx}}hk + \left[\frac{f_{xy}k}{f_{xx}}\right]^2\right) + f_{yy}k^2 - \frac{f_{xy}{}^2k^2}{f_{xx}} \\
&= f_{xx}\left(h + \frac{f_{xy}k}{f_{xx}}\right)^2 + \frac{k^2}{f_{xx}}\left(f_{xx}f_{yy} - f_{xy}{}^2\right)
\end{aligned}$$

Thus, $D^2_{\vec{u}} f = f_{xx}\left(h + \frac{f_{xy}k}{f_{xx}}\right)^2 + \frac{k^2}{f_{xx}}\left(f_{xx}f_{yy} - f_{xy}{}^2\right)$. Consequently, if $D = f_{xx}f_{yy} - f_{xy}{}^2 > 0$ and $f_{xx} > 0$, then $D^2_{\vec{u}} f > 0$ for all unit vectors $\vec{u}$. Thus, any plane that passes through $z = f(x, y)$ and contains the point $(a, b, f(a, b))$ and is perpendicular to the *xy*-plane will result in a cross-section with $z = f(x, y)$ that is concave up. Therefore, $(a, b, f(a, b))$ is a minimum point. If $D = f_{xx}f_{yy} - f_{xy}{}^2 > 0$ and $f_{xx} < 0$, then $D^2_{\vec{u}} f < 0$ for all unit vectors $\vec{u}$, and the argument is similar that $(a, b, f(a, b))$ is a maximum point.

Now suppose that $D = f_{xx} f_{yy} - f_{xy}{}^2 < 0$, and recall that $D^2{}_{\vec{u}} f = f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2$.

Suppose $f_{xx} \neq 0$, and note that

$$f_{xx}D_{\vec{u}}{}^2 f = f_{xx}\left(f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2\right) = f_{xx}{}^2h^2 + 2f_{xx}f_{yy}hk + f_{xx}f_{yy}k^2$$

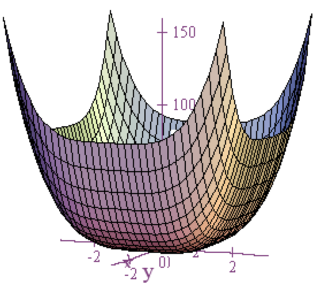
$$= f_{xx}{}^2h^2 + 2f_{xx}f_{xy}hk + f_{xy}{}^2k^2 + f_{xx}f_{yy}k^2 - f_{xy}{}^2k^2 = \left(f_{xx}h + f_{xy}k\right)^2 + \left(f_{xx}f_{yy} - f_{xy}{}^2\right)k^2.$$

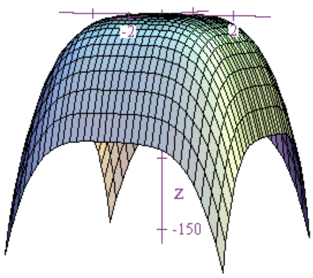
Hence, $f_{xx}D_{\vec{u}}{}^2 f > 0$ when $h \neq 0$ and $k = 0$, and $f_{xx}D_{\vec{u}}{}^2 f < 0$ when $f_{xx}h + f_{xy}k = 0$ and $k \neq 0$. This implies that $D_{\vec{u}}{}^2 f$ is positive in one direction and negative in another, thus implying that the graph of $z = f(x,y)$ is concave-up in one direction at $(a,b,f(a,b))$ and concave-down in another direction. Therefore, $(a,b,f(a,b))$ is a saddle point. Also, if $f_{yy} \neq 0$, then a similar argument may be used to arrive at the same conclusion that $(a,b,f(a,b))$ is a saddle point. One may ask, though, in the argument above how it is that we know that we can have both $f_{xx}h + f_{xy}k = 0$ and $k \neq 0$. Well, if $k \neq 0$, then $f_{xx}h + f_{xy}k = 0 \Rightarrow \dfrac{h}{k} = -\dfrac{f_{xy}}{f_{xx}}$. However, since (h,k) is a point on the unit circle, we have that $\dfrac{h}{k} = \cot\theta$ where θ is the angle made with the positive *x*-axis. Furthermore, since $\cot\theta$ takes on every real number value as θ goes from 0 to 2π, it certainly, at some point, takes on the value $-\dfrac{f_{xy}}{f_{xx}}$. And from this we get that $f_{xx}h + f_{xy}k = 0$.

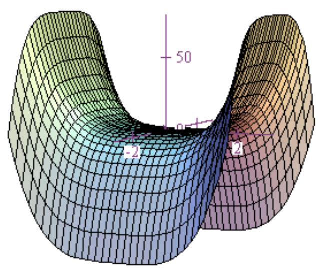
Now, suppose again that $D = f_{xx} f_{yy} - f_{xy}{}^2 < 0$, and recall that

$D^2{}_{\vec{u}} f = f_{xx}h^2 + 2 f_{xy}hk + f_{yy}k^2$. If $f_{xx} = 0 = f_{yy}$, then $D^2{}_{\vec{u}} f = f_{xx}h^2 + 2 f_{xy}hk + f_{yy}k^2$ reduces to $D^2{}_{\vec{u}} f = 2 f_{xy}hk$ and $D = -f_{xy}{}^2 < 0$. Hence, $f_{xy} \neq 0$, and $D_{\vec{u}}{}^2 f$ will have different signs for the unit vectors $\vec{u}_1 = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$ and $\vec{u}_2 = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$. Therefore, $(a, b, f(a,b))$ is a saddle point.

The final thing we need to show is that if $D = 0$, then the test is inconclusive. This can be done simply by examining the graphs of $z = x^4 + y^4$, $z = -x^4 - y^4$, and $z = x^4 - y^4$. You can easily show that each of these functions has $(0,0)$ as a critical point, and each function results in $D = 0$. However, the graph of the first function displays a local minimum at the critical point, the graph of the second shows a local maximum, and the third graph has a saddle point at $(0,0)$. Thus, these three examples show that anything can happen when $D = 0$.





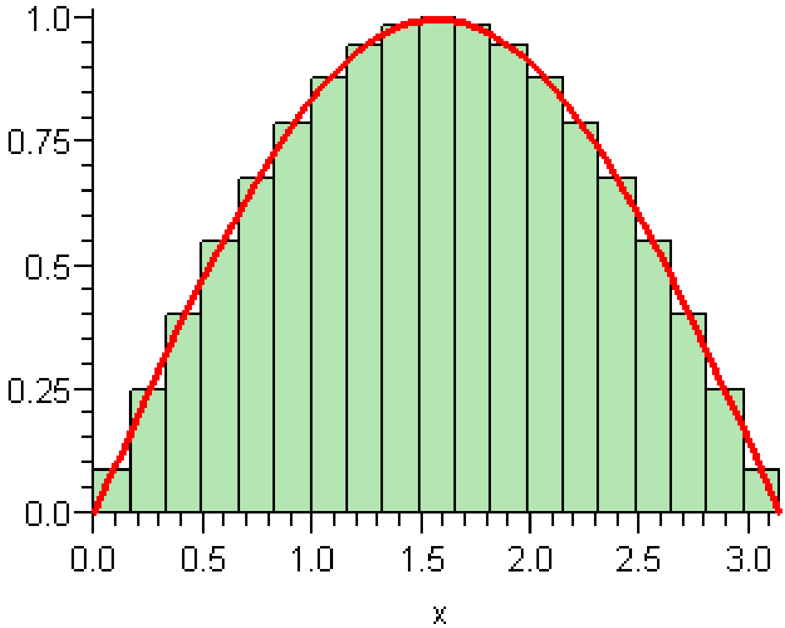


By golly, I think we're done! ■

CHAPTER 8

INTEGRALS

You'll undoubtedly recall that you first learned about integrals by trying to figure out how to find the area under a curve.



The strategy was to subdivide our interval from a to b into a series of n subintervals of width $\Delta x = \dfrac{b-a}{n}$, evaluate our function at a point in each subinterval in order to get a height for a rectangle, add up the areas of the rectangles, and use that as an approximation for the area under the curve. Our expectation, of course was that as the number of subintervals increase, the approximation would get better and better. Thus, for the case in which we had $f(x) \ge 0$, we wrote

$$Area = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum f(x)\cdot \Delta x = \lim_{\Delta x\to 0} \sum f(x)\cdot \Delta x$$

And to help with computational matters, we soon discovered two versions of the Fundamental Theorem of Calculus.

<u>The Fundamental Theorem of Calculus:</u> Let $y = f(x)$ be continuous on the interval $a \le x \le b$. Then,

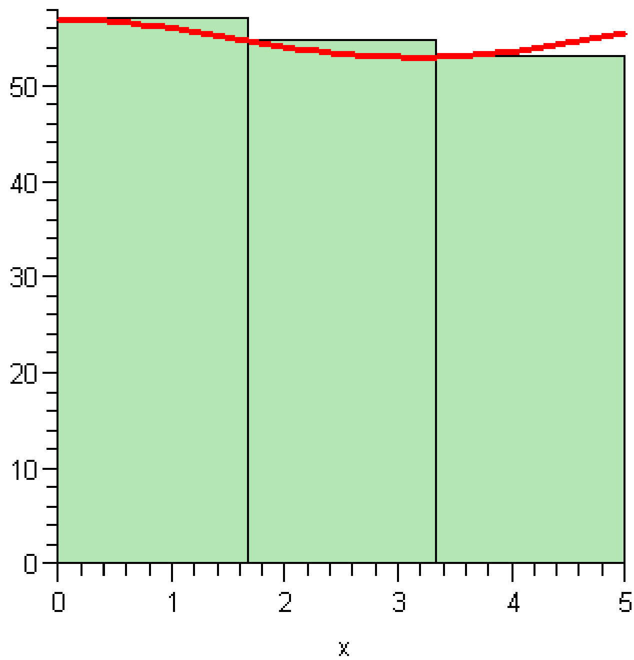
1. (The Derivative of the Integral) If $A(x) = \int_a^x f(u)\,du$, then $\dfrac{dA}{dx} = f(x)$.
2. (The Integral of the Derivative) If $F(x)$ is any antiderivative of $f(x)$, then

   $\int_a^b f(x)\,dx = F(b) - F(a)$.

As a consequence of this theorem, evaluating integrals boiled down to an often simple process of finding antiderivatives of familiar functions, and much of single variable calculus is devoted to the abstract art of finding such antiderivatives. Nonetheless, one also encounters more applied problems such as this one.

Problem: Suppose you are driving down a highway, and your speed fluctuates in such a way that at time t hours, your speed is $f(t) = 55 + 2\cos(t)\frac{miles}{hour}$. Find the distance you have traveled after 5 hours.

The apparently tricky thing in this problem is that our speed is variable. In fact, over time your speed will drift from a low of 53 miles per hour to 57 miles per hour. However, in the diagram below, we try to approximate the area under the curves using rectangles.



What becomes apparent from these rectangles is that since we are using a constant speed for the height of each rectangle, that means that we can use our familiar $\text{distance} = \text{rate} \times \text{time}$ formula to get the distance traveled over each subinterval. Furthermore, this distance corresponds to the area of the related rectangle. And additionally, it's easy to see that if we increase the number of rectangles, then we get a better approximation for the actual distance traveled when our speed is variable. However, increasing the number of rectangles also results in a better approximation of the area under the curve. Consequently, we can conclude that when we have a variable speed given by a function $f(t)$ on an interval $a \le t \le b$, then the distance traveled is given by $\int_a^b f(t)\,dt$. There is, however, one more important thing we can learn from this example, and that is how we deal with units in most integrals related to the real world. If we write in the units associated with $f(t)$ and *t*, then we can express our integral as follows.

$$\lim_{\Delta t \to 0} \sum f(t) \frac{miles}{hour} \cdot \Delta t\, hours = \left(\lim_{\Delta t \to 0} \sum f(t) \cdot \Delta t \right) \frac{miles}{hour} hours = \int_a^b f(t)\,dt \;\; hours$$

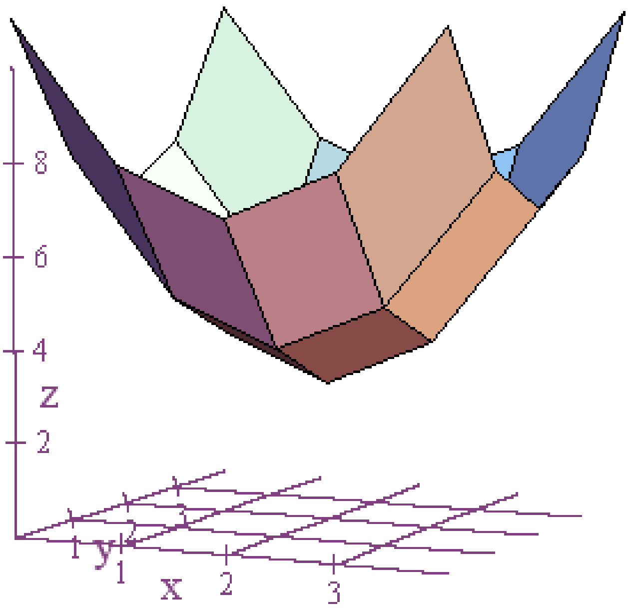
In other words, when units are involved, we can write the units on our integral as the product of the units on our function with the units on our input variable.

If we now have a function of two variables such as $z = f(x, y)$, we can define integrals in a similar way. In particular, suppose $z = f(x, y) = (x-2)^2 + (y-2)^2 + 3$,

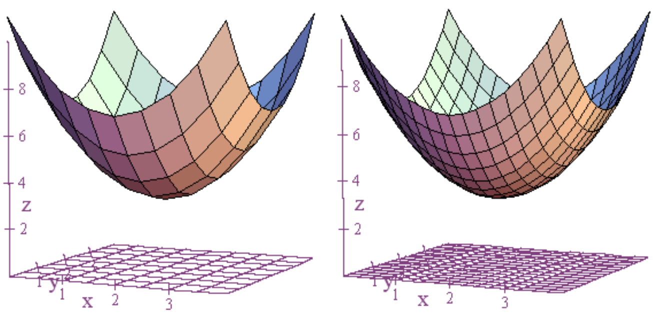
$0 \le x \le 4$, and $0 \le y \le 4$. Suppose also that the units of feet are attached to all three variables, *x*, *y*, and *z*. In this case, the expression $\sum f(x,y)\Delta x \Delta y$ will give us an approximation of the volume beneath our surface and above the *xy*-plane. Furthermore, if we multiply the output units by both of the input units, then we get units of cubic feet for our approximation. Additionally, in the above formula, since $\Delta x \Delta y$ is going to represent an element of area in the *xy*-plane, we often represent this area by ΔA. And finally, if we take the limit of the above expression as both Δx and Δy go to zero, then our volume approximation converges to the exact volume, and we call the result of such a limit a double integral since we have two input variables. We often write it like this,

$$\iint_R f(x,y)\,dA = \lim_{\Delta x, \Delta y \to 0} \sum f(x,y)\,\Delta A$$

In this formula, *R* is used to represent the region in the *xy*-plane that we are integrating over. The picture below uses $z = f(x,y) = (x-2)^2 + (y-2)^2 + 3$, $0 \le x \le 4$, and $0 \le y \le 4$, and it illustrates much of what we are talking about.



The grid in the *xy*-plane has the *x*-interval subdivided into four equal subintervals, and the *y*-interval is likewise subdivided into four subintervals. That gives a grid that contains sixteen rectangles total. Also, you see that if you plot the points $(x, y, f(x, y))$ above the corner points of each of our rectangles and if you then connected the dots, then you will get a graph that approximates the surface graph for $z = f(x, y) = (x-2)^2 + (y-2)^2 + 3$. Now suppose that for each of our sixteen rectangles we take the corner point with the smallest values for *x* and *y* and we use the corresponding *z*-value as the height of a box that we erect over our rectangle. Then the sum of the volume of the boxes will approximate the volume under our surface and above the *xy*-plane. Furthermore, as we take finer and finer subdivisions of both our *x*-interval and our *y*-interval, we get not only better approximations for the volume, but also a graph that better approximates our actual surface. Below is the grid and graph that results from subdividing each interval into eight parts, and this is followed by a picture where each side of the grid is subdivided into sixteen parts. As you can see, this improves the quality of the graph, and it will also improve any volume approximations.



The question we want to address now concerns how we go about actually evaluating $\iint_R f(x,y)\,dA = \lim_{\Delta x,\Delta y \to 0} \sum f(x,y)\,\Delta A$. The answer, fortunately, is given by a result known as Fubini's Theorem. When I was a sophomore in college taking multivariable calculus for the first time, my teacher described Fubini's Theorem as saying that if you are doing integration of functions of several variables and you get any answer at all, then it must be right! Well, Fubini's Theorem is not that generous, but close!

<u>Fubini's Theorem:</u> If $z = f(x,y)$ is continuous on the rectangle

$R = \{(x,y):\ a \le x \le b,\ c \le y \le d\}$, then $\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dydx = \int_c^d\int_a^b f(x,y)\,dxdy$.

These last two integrals are called iterated integrals, and it simply means that you first evaluate the inner integral, and then you take that result and integrate again. What Fubini's Theorem then tells us is that it doesn't matter whether you integrate first with respect to *x* or first with respect to *y*. The end result will be the same. Also, just as we treat other variables as constants when we differentiate with respect to *x*, we do the same when integrating with respect to *x*. In other words, whenever you integrate with respect to one particular variable, you always treat the other variables as constants. Now let's use $z = f(x,y) = (x-2)^2 + (y-2)^2 + 3$, $0 \le x \le 4$, and $0 \le y \le 4$ as an example. We'll find the volume under the surface in two ways, first by integrating with respect to *y* and then *x*, and then by doing the opposite order. Also, since both *x* and *y* vary from 0 to 4, we'll have the same limits of integration on both integrals both times. Here we go!

$$\iint_R f(x,y)\,dA = \int_0^4\int_0^4 \left[(x-2)^2 + (y-2)^2 + 3\right]dydx = \int_0^4 \left[\left.(x-2)^2 y + \frac{(y-2)^3}{3} + 3y\right|_0^4\right] dx$$

$$= \int_0^4 \left[(x-2)^2 4 + \frac{8}{3} + 12\right] dx = \left.\frac{4(x-2)^3}{3} + \frac{8x}{3} + 12x\right|_0^4 = \frac{32}{3} + \frac{32}{3} + 48 = \frac{208}{3}$$

Similarly,

$$\iint_R f(x,y)\,dA = \int_0^4\int_0^4 \left[(x-2)^2+(y-2)^2+3\right]dxdy = \int_0^4 \left[\frac{(x-2)^3}{3}+(y-2)^2x+3x\right]\Bigg|_0^4 dx$$

$$= \int_0^4 \left[\frac{8}{3}+(y-2)^2 4+12\right]dy = \frac{8y}{3}+\frac{4(y-2)^3}{3}+12y\Bigg|_0^4 = \frac{32}{3}+\frac{32}{3}+48 = \frac{208}{3}$$

We get the same result no matter what order we do our integration in, and that's exactly what Fubini's Theorem says ought to happen. Now let's look at a proof of sorts. It's not a full-blown, fully accurate proof. Instead, it's more of an outline of how the proof goes with the messier details omitted. Thus, I'll just call it an argument. The argument depends, though, on familiar properties of real numbers such as the commutative property of addition. In other words, when we are adding together the terms of the sum $\sum f(x,y)\Delta A = \sum f(x,y)\Delta y\Delta x$, we can add those terms up in any order we want. And as we take limits as $\Delta x, \Delta y \to 0$, this will result in different integral expressions that, nonetheless, give us the same overall result. Throughout I try to avoid cluttering up my expressions with subscripts, but I will use *i* to represent the i^{th} subinterval along the *x*-axis and *j* to represent the j^{th} subinterval along the *y*-axis. Here goes!

Argument: As any idiot can plainly see,

$$\iint_R f(x,y)\,dA \approx \sum_{i,j} f(x,y)\Delta A = \sum_i \left(\sum_j f(x,y)\Delta y \right)\Delta x = \sum_j \left(\sum_i f(x,y)\Delta x \right)\Delta y$$

Thus,

$$\iint_R f(x,y)\,dA = \lim_{\Delta x,\Delta y\to 0} \sum_{i,j} f(x,y)\Delta A = \lim_{\Delta x\to 0} \sum_i \left(\lim_{\Delta y\to 0} \sum_j f(x,y)\Delta y \right)\Delta x = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

Similarly,

$$\iint_R f(x,y)\,dA = \lim_{\Delta x,\Delta y\to 0} \sum_{i,j} f(x,y)\Delta A = \lim_{\Delta y\to 0} \sum_j \left(\lim_{\Delta x\to 0} \sum_y f(x,y)\Delta x \right)\Delta y = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

Consequently,

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dydx = \int_c^d \int_a^b f(x,y)\,dxdy \qquad \blacksquare$$

And now, here's another example of Fubini's Theorem in practice.

$$\int_0^1 \int_2^4 xy^2\,dydx = \int_0^1 \left(\frac{xy^3}{3} \right)\Bigg|_2^4 dx = \int_0^1 \left(\frac{64x}{3} - \frac{8x}{3} \right) dx = \int_0^1 \frac{56x}{3}\,dx = \frac{28x^2}{3}\Bigg|_0^1 = \frac{28}{3} - \frac{0}{3} = \frac{28}{3}$$

Likewise,

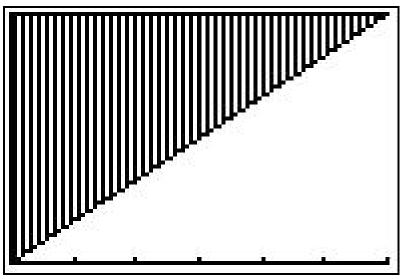
$$\int_2^4 \int_0^1 xy^2\,dxdy = \int_2^4 \left(\frac{x^2y^2}{2} \right)\Bigg|_0^1 dy = \int_2^4 \left(\frac{y^2}{2} - \frac{0}{2} \right) dy = \int_2^4 \frac{y^2}{2}\,dy = \frac{y^3}{6}\Bigg|_2^4 = \frac{64}{6} - \frac{8}{6} = \frac{56}{6} = \frac{28}{3}$$

Often times we change the order of integration simply because the integral is impossible for us to do by hand in one order, but very easy in the other. Here's one such example. We start with $\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\,dydx$. Well, I'm sorry, but I don't know an antiderivative, with respect to *y*, for $x\sqrt{y^3+1}$. Hence, let's see if we can rewrite this so that we integrate with respect to *x* first. If we do, however, then we'll generally also need to make some adjustments to our limits of integration. Now the limits we have on these integrals tell us that,

$0 \le x \le 6$ and

$\frac{x}{3} \le y \le 2$

If we graph this region, we get something like the following.



What we need to do is express this same region in terms of inequalities such that this time *y* varies from one number to another and *x* varies from one function of *y* to another function of *y*. By doing this, we will determine our limits of integration for the new integral. Well, it's pretty clear (hopefully!) that we want $0 \le y \le 2$. Also, if we take our equation for the diagonal line, $y = \frac{x}{3}$, and solve it for *x*, we get $x = 3y$.

Therefore, our second inequality should be $0 \le x \le 3y$. Hence, the new integral is

$\int_0^2 \int_0^{3y} x\sqrt{y^3+1}\,dxdy$, and this one is pretty easy to do.

$$\int_0^2 \int_0^{3y} x\sqrt{y^3+1}\,dxdy = \int_0^2 \left.\frac{x^2\sqrt{y^3+1}}{2}\right|_0^{3y} dy = \int_0^2 \frac{9y^2\sqrt{y^3+1}}{2}dy = \int_1^9 \frac{3u^{1/2}}{2}du$$
$$= 9^{3/2} - 1^{3/2} = 27 - 1 = 26$$

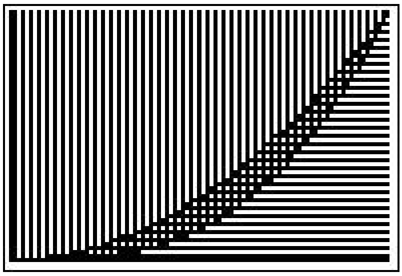
Another way we can use double integrals is to find areas. To do this, just think of your integrand as being $f(x,y) = 1$. In other words, if you have a solid whose base is some region *R*, but the height of that solid is 1, then the volume will numerically be the same as the area of the base. That's what we're doing here!

<u>Example 1:</u> Use a double integral to find the area of the region between the curves $y = x^2$ and $y = x^3$ from $x = 0$ to $x = 1$.

<u>Solution:</u> On this interval, the value of x^2 will be greater than that of x^3. Thus, our function and intervals for *x* and *y* are,

$$z = f(x,y) = 1$$
$$0 \le x \le 1$$
$$x^3 \le y \le x^2$$

Also, here is a picture of the region *R* whose area we want to find,



Now all we need to do is evaluate the integral.

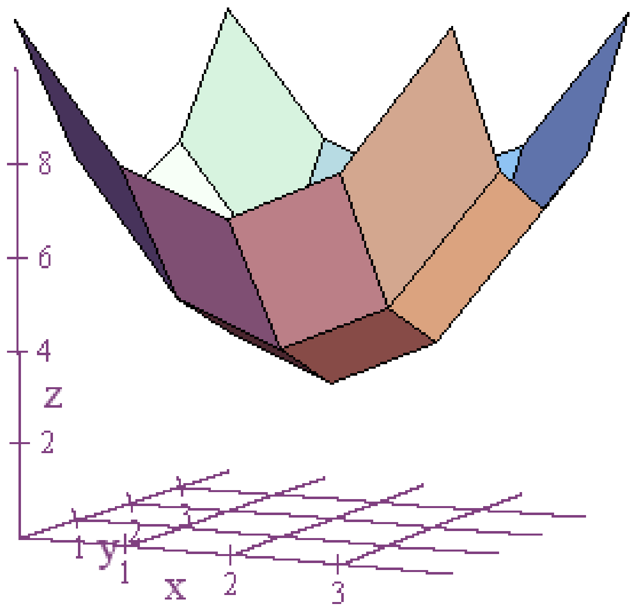
$$Area = \iint_R dA = \int_0^1 \int_{x^3}^{x^2} dy dx = \int_0^1 y \Big|_{x^3}^{x^2} dx = \int_0^1 \left(x^3 - x^2\right) dx = \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Bigg|_0^1$$
$$= \left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{0}{3} - \frac{0}{4}\right) = \frac{4}{12} - \frac{3}{12} = \frac{1}{12}$$

We can also reverse the order of integration here and integrate first with respect to x. If we do that, then we have to find a numerical interval over which y varies and a pair of functions of y that x will lie between. In this case, it looks like we want $0 \le y \le 1$ and $\sqrt{y} \le x \le \sqrt[3]{y}$. Thus,

$$Area = \iint_R dA = \int_0^1 \int_{\sqrt{y}}^{\sqrt[3]{y}} dx dy = \int_0^1 x \Big|_{\sqrt{y}}^{\sqrt[3]{y}} dy = \int_0^1 \left(y^{1/3} - y^{1/2}\right) dy = \left(\frac{3y^{4/3}}{4} - \frac{2y^{3/2}}{3}\right)\Bigg|_0^1$$
$$= \left(\frac{3}{4} - \frac{2}{3}\right) - \left(\frac{0}{4} - \frac{0}{3}\right) = \frac{9}{12} - \frac{8}{12} = \frac{1}{12}$$

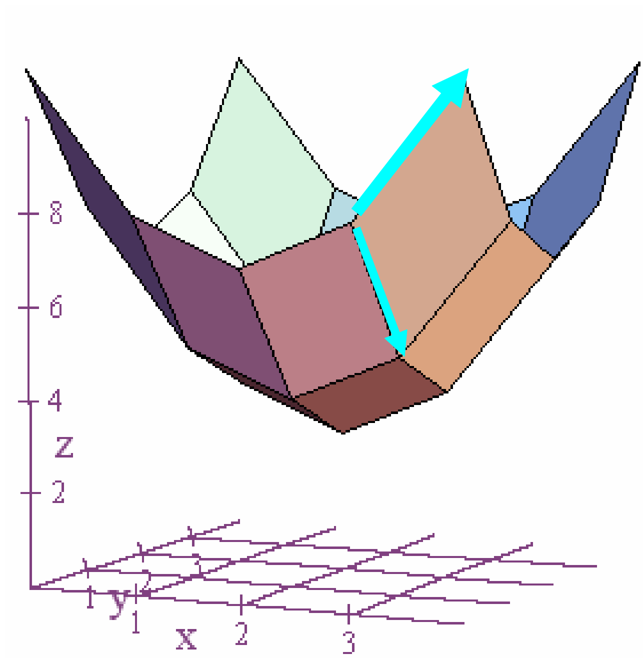
And we get the same answer! Exactly as Fubini's Theorem predicts.

Another application of double integrals is to compute surface area. For example, let's take a function we looked at earlier, $z = f(x,y) = (x-2)^2 + (y-2)^2 + 3$ with $0 \le x \le 4$ and $0 \le y \le 4$. If we look closely at the wireframe graph below, we might notice that above each rectangle in our region R in the xy-plane there lies a corresponding parallelogram whose area approximates the surface area above that rectangle.

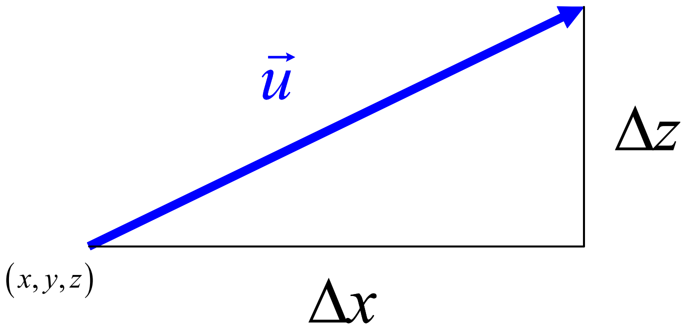


If we find the areas of these parallelograms and add them up, then the sum will approximate the area of our surface. Furthermore, as we partition our region R into smaller and smaller subrectangles, our approximation of the surface area should become more accurate. The question now is how do we find the areas of our parallelograms? And the answer is, vectors! Recall that if a have a parallelogram defined by two vectors $\vec{u}$ and $\vec{v}$, then the area of that parallelogram is given by

$\|\vec{u}\times\vec{v}\|$. Below is our wireframe drawing again, but this time with two vectors added to one of the parallelograms on the surface.



If we designate, on our rectangle in the *xy*-plane, the corner point with the smallest coordinates as (x, y), then the two adjacent corner points will have coordinates $(x+\Delta x, y)$ and $(x, y+\Delta y)$. Thus, I claim we can define our vectors $\vec{u}$ and $\vec{v}$ as $\vec{u}\approx\Delta x\,\hat{i}+0\,\hat{j}+\frac{\partial f}{\partial x}\Delta x\,\hat{k}$ and $\vec{v}\approx 0\,\hat{i}+\Delta y\,\hat{j}+\frac{\partial f}{\partial y}\Delta y\,\hat{k}$. Now the question is how do we get this? Well, let's answer for the vector $\vec{u}$ by looking at a typical diagram below.



In this diagram, our vector $\vec{u}$ lies in a plane parallel to the *xz*-plane, and hence, as we traverse the length of $\vec{u}$ there is a change in *x* and a change in *z*, but no change in *y*. Also, notice that $\vec{u} = \Delta x\,\hat{i} + \Delta z\,\hat{k}$ and that the slope of $\vec{u}$ is $\dfrac{\Delta z}{\Delta x}$. Realize, too, that if Δx is small, then $\vec{u}$ will approximate a tangent vector at (x, y, z) pointing in the direction of the positive *x*-axis, and thus, it's slope is also approximately equal to $\dfrac{\partial f}{\partial x}$ evaluated at (x, y, z). Hence, we get that $\dfrac{\Delta z}{\Delta x} \approx \dfrac{\partial f}{\partial x} \Rightarrow \Delta z \approx \dfrac{\partial f}{\partial x}\Delta x$, and $\vec{u} \approx \Delta x\,\hat{i} + 0\,\hat{j} + \dfrac{\partial f}{\partial x}\Delta x\,\hat{k}$. A similar argument with respect to *y* shows that $\vec{v} \approx 0\,\hat{i} + \Delta y\,\hat{j} + \dfrac{\partial f}{\partial y}\Delta y\,\hat{k}$, and both of these approximations improve as $\Delta x, \Delta y \to 0$. Consequently,

$$\vec{u} \times \vec{v} \approx \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta x & 0 & \dfrac{\partial f}{\partial x}\Delta x \\ 0 & \Delta y & \dfrac{\partial f}{\partial y}\Delta y \end{vmatrix} = \left(-\frac{\partial f}{\partial x}\Delta x\Delta y\right)\hat{i} - \left(\frac{\partial f}{\partial y}\Delta x\Delta y\right)\hat{j} + \left(\Delta x\Delta y\right)\hat{k}$$

And,

$$\|\vec{u}\times\vec{v}\| \approx \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 \Delta x^2 \Delta y^2 + \left(\frac{\partial f}{\partial y}\right)^2 \Delta x^2 \Delta y^2 + \Delta x^2 \Delta y^2} = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \cdot \Delta x \Delta y$$

Therefore,

$$Surface\ Area = \lim_{\Delta x, \Delta y \to 0} \sum_{i,j} \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \cdot \Delta x \Delta y$$

$$= \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dxdy = \iint_R \left(\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \right) dA$$

Also, if we let *S* denote the surface we are integrating over, and if we denote an element of area on the surface by ΔS, then what we've also shown above is that

$\Delta S \approx \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \cdot \Delta A$, and hence,

$$Surface\ Area = \iint_S dS = \iint_R \left(\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \right) dA$$

Consequently, we're really turning a double integral over a surface *S* into a more manageable double integral over a region *R* in the plane. Additionally, remember the formula $dS = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dA$ because we'll see it again in the chapter on Green's Theorem, Stokes' Theorem, and the Divergence TheoremI.

Finally, we have a very nice formula for surface area! Unfortunately, in practice this often leads to something that is hard to integrate, and that will be the case with $z = f(x, y) = (x-2)^2 + (y-2)^2 + 3$ with $0 \le x \le 4$ and $0 \le y \le 4$. In this instance,

$$\begin{aligned}
z_x &= 2(x-2) = 2x-4 \\
z_y &= 2(y-2) = 2y-4 \\
{z_x}^2 &= 4x^2 - 16x + 16 \\
{z_y}^2 &= 4y^2 - 16y + 16
\end{aligned}$$

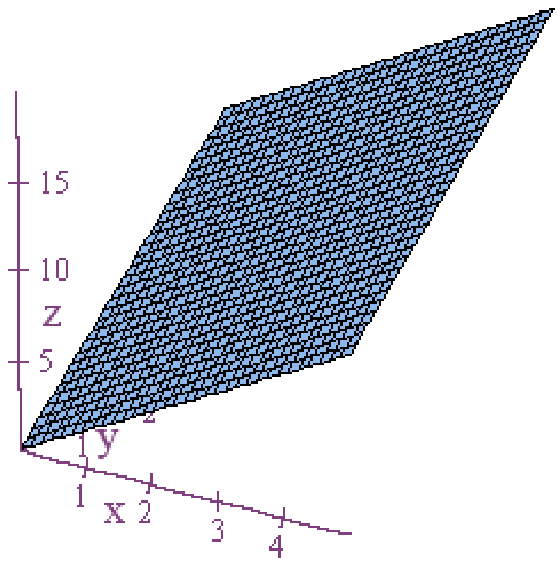
Hence,

$$\mathit{Surface\ Area} = \iint_R \sqrt{4x^2 + 4y^2 - 16x - 16y + 33}\, dA$$

Have fun doing that one! Often times with integrals such as these we have to evaluate them numerically. However, later in this chapter we'll see how to do a similar integral by making a change of variables. In fact, the only reason we're not going to do it at this point is because our region *R* doesn't have the appropriate shape for the kind of change we have in mind. For now, though, take it on faith that if I plug the above integral into *MAPLE* software and ask for a numerical approximation of the answer, then I get back that the surface area is approximately 52 square units. Close enough for government work!

We should probably do at least one example that does lead to something we can integrate, so let's try this one.

Example 2: Find the surface area of the plane defined by $z = 2x + 3y$ where $0 \le x \le 5$ and $0 \le y \le 4$.



Solution: The necessary partial derivative computations are,

$$z_x = 2 \Rightarrow z_x^{\;2} = 4$$
$$z_y = 3 \Rightarrow z_y^{\;2} = 9$$

Thus,

$$Surface\ Area = \int_0^5\int_0^4 \sqrt{14}\,dydx = \int_0^5 y\sqrt{14}\Big|_0^4 dx = \int_0^5 4\sqrt{14}\,dx = 4x\sqrt{14}\Big|_0^5 = 20\sqrt{14} \approx 74.833$$

We can verify this result by taking the vectors $\vec{u}$ and $\vec{v}$ that define this parallelogram portion of the plane, and then calculate $\left\|\vec{u}\times\vec{v}\right\|$. Thus, $\vec{u} = 5\hat{i} + 10\hat{k}$ and $\vec{v} = 4\hat{j} + 12\hat{k}$.

Hence,

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 0 & 10 \\ 0 & 4 & 12 \end{vmatrix} = -40\hat{i} - 60\hat{j} + 20\hat{k}$$

And,

$$\|\vec{u} \times \vec{v}\| = \sqrt{1600 + 3600 + 400} = \sqrt{5600} = \sqrt{400 \cdot 14} = 20\sqrt{14} \approx 74.833$$

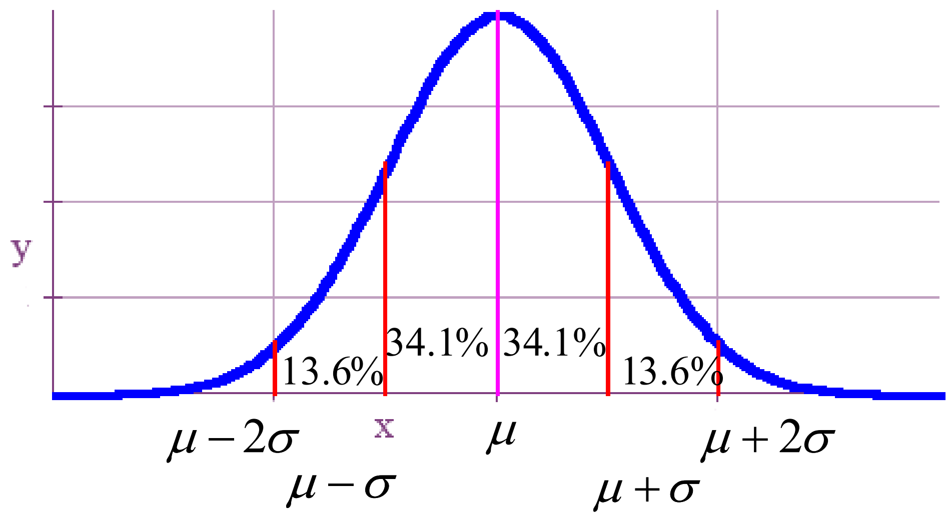
I love it when things work out!

We now want to look at a very different application of double integrals, but first we need to define what we mean by a probability density function (also called a probability distribution function). If we are dealing with a function of one variable, $p = p(x)$, then this function could be a probability density function if the following two conditions are met,

1. $p(x) \geq 0$ for all x

2. $\int_{-\infty}^{\infty} p(x)\,dx = 1$

The most widely known example of a probability density function of one variable is the one that gives rise to the bell-shaped curve known as the normal curve or normal distribution. It's defined by $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ where μ is the mean or average value of the distribution and σ is the standard deviation, a measure of how spread out the values of the distribution are. A typical normal curve looks like the following.



Also, as this picture suggests, we find probabilities in a probability density function by calculating the area under the curve. Thus, for example, in the normal distribution above there is a 34.1% chance that a score will fall between the mean and one standard deviation above the mean, and there is a 13.6% chance a score will be between $\mu + \sigma$ and $\mu + 2\sigma$.

When two variables are involved, a probability density function is also called a joint density function, and the two conditions a function $p(x, y)$ must follow in order to be a joint density function are,

1. $p(x,y) \ge 0$ for all x and y

2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dydx = 1$

Given that we have a joint density function, to find the probability that $a \le x \le b$ and $c \le y \le d$ m we evaluate $P(a \le x \le b,\ c \le y \le d) = \int_a^b\int_c^d p(x,y)\,dydx$. Here's an easy example.

Example 3: Given the joint density function $p(x,y) = \begin{cases} x+y & \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \textit{otherwise} \end{cases}$, find the probability that $0 \le x \le \frac{1}{2}$ and $0 \le y \le \frac{1}{2}$.

Solution: Probability = $P\begin{pmatrix} 0 \le x \le \frac{1}{2} \\ 0 \le y \le \frac{1}{2} \end{pmatrix} = \int_0^{1/2}\int_0^{1/2} (x+y)\,dydx = \int_0^{1/2} \left(xy + \frac{y^2}{2} \right)\Bigg|_0^{1/2} dx$

$= \int_0^{1/2} \left(\frac{x}{2} + \frac{1}{8} \right) dx = \left(\frac{x}{2} + \frac{1}{8} \right)\Bigg|_0^{1/2} = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$. ■

Next, I want to show you a very useful theorem. It basically says that if you have two probability density functions of one variable, then it's very easy to construct a joint density function from them. For example,, both height and weight for adult men tend to be normally distributed. Using the theorem we will prove below, we can easily construct a corresponding joint density function for answering questions such as what is the probability that an adult male has a height between five feet and six feet and a weight between 180 pounds and 200 pounds?

Theorem: If $p(x)$ and $q(y)$ are both probability density functions of one variable, then $f(x,y) = p(x)q(y)$ is a joint density function.

Proof: It suffices to show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x)q(y)\,dydx = 1$. Clearly, though, since functions of one variable may be treated as constants when integrated with respect to another variable, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x)q(y)\,dydx = \int_{-\infty}^{\infty} p(x)\left(\int_{-\infty}^{\infty} q(y)\,dy\right)dx = \left(\int_{-\infty}^{\infty} q(y)\,dy\right)\left(\int_{-\infty}^{\infty} p(x)\,dx\right) = 1\cdot 1 = 1\,. \quad \blacksquare$$

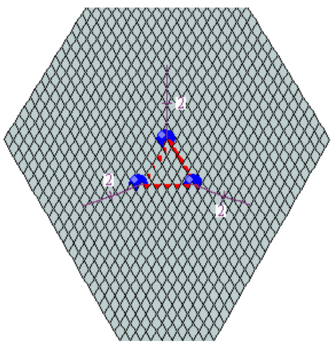
This proof also shows us how to evaluate a joint density function that has been constructed from two probability density functions of a single variable. We merely multiply two individually calculated probabilities together. For example, let's suppose that adult men have an average height of 5.75 feet (5 feet, 9 inches) with a standard deviation of 3 inches (0.25 feet), and that the average weight of an adult male is 190 pounds with a standard deviation of 10 pounds. Then, technically speaking, the probability that an adult male has a height between 5 feet and 6 feet is $\int_5^6 \frac{1}{0.25\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-5.75}{0.25}\right)^2} dx$, and the probability that an adult male has a weight between 180 pounds and 200 pounds is $\int_{180}^{200} \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-190}{10}\right)^2} dy$. Consequently, the joint probability that an adult male has a height between 5 feet and 6 feet and a weight between 180 pounds and 200 pounds is

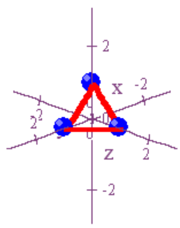
$\int_5^6 \frac{1}{0.25\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-5.75}{0.25}\right)^2} dx \cdot \int_{180}^{200} \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-190}{10}\right)^2} dy$. For those who know a little probability theory, this product should come as no surprise because whenever we are finding the probability of two events joined by the word "and," we generally multiply simpler probabilities together. In this case, though, we can't find simple expressions for the antiderivatives of our integrands, and so we have to evaluate these integrals numerically. Fortunately, these days even a TI-83 or TI-84 calculator can easily do the math. Using the normal cumulative distribution key on this calculator, we get that the probability that the height is from 5 to 6 feet is approximately 0.84, and the probability that the weight is in the specified range is about 0.68. Thus, the probability that both these events occur is $0.84 \cdot 0.68 \approx 0.57$.

Now let's increase the level of complexity one notch above that of double integrals. Let's talk about triple integrals. Mathematically speaking, a triple integral is just an integral that is done with respect to a volume in three dimensions as opposed to an area in two dimensions that we integrate over when doing double integrals. Thus, in a triple integral, an element of volume is defined by a product of the change in *y* times the change in *x* times the change in *z*. Symbolically, we can write this as change in volume $=\Delta V=\Delta x\Delta y\Delta z$, and in differential form we write this as $dV = dxdydz$. Now let's suppose that we have a function of three variables such as $w = f(x, y, z)$ and that we want to integrate this function over a solid region *V* in three dimensional space. Then we write this integral as $\iiint_V f(x, y, z)\,dV$. Now we have to ask, how do we evaluate this integral? Fortunately, Fubini's Theorem can be proven for this higher dimensional case, and thus, we can write the integral as three separate iterated integrals. In particular, suppose the region *V* can be described by the inequalities $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, and $h_1(x, y) \le z \le h_2(x, y)$. Then $\iiint_V f(x, y, z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)\,dzdydx$. By Fubini's Theorem, this triple integral can be written as an iterated integral in six different ways corresponding to the six different permutations we can make of the variables *x*, *y*, and *z*. Of course, some orders may lead to an easy integration while others lead to expressions that it's difficult or impossible to find simple antiderivatives for. Furthermore, if our integrand is equal to 1, then the result of our triple integral is volume $= \iiint_V dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} dzdydx$. Now let's look at a couple of example problems involving triple integrals.

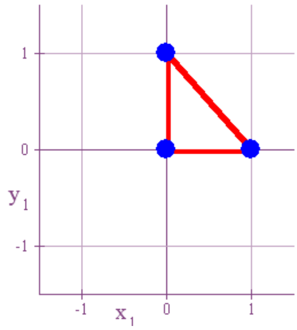
<u>Example 4:</u> Find the volume of the solid in the first octant that is bounded below by the *xy*-plane and above by the plane $x+y+z=1$.

<u>Solution:</u> The whole trick to triple integrals is figuring out how to describe your region of integration in terms of intervals involving the variables *x*, *y*, and *z*. A picture can be very helpful, but it still takes a certain amount of practice and ingenuity.





In this case, though, our solid is bounded above by the plane $x+y+z=1$ and below by the *xy*-plane. This suggests that if we rewrite $x+y+z=1$ as $z=1-x-y$, then we can say throughout that *z* varies from $z=0$ to $z=1-x-y$. Consequently, all we need to do now is to describe the bottom part of our solid in terms of *x* and *y*. In two dimensions, our bottom part looks like this.



The corner points are at $(0,0)$, $(0,1)$, and $(1,0)$. Additionally, the line through $(0,1)$ and $(1,0)$ can be described by $y=-x+1$. Thus, we can describe the enclosed triangular region by the inequalities $0\le x\le 1$ and $0\le y\le -x+1$. We're now ready to list all our inequalities together and to find the volume of our solid.

$$\begin{aligned}&0\le x\le 1\\&0\le y\le -x+1\\&0\le z\le 1-x-y\end{aligned}$$

$$\text{Volume } =\iiint_V dV=\int_0^1\int_0^{-x+1}\int_0^{1-x-y} dzdydx=\int_0^1\int_0^{-x+1}(1-x-y)\,dydx$$

$$=\int_0^1\left(y-xy-\frac{y^2}{2}\right)\Bigg|_0^{-x+1}dx=\int_0^1\left(-x+1-x(-x+1)-\frac{(-x+1)^2}{2}\right)dx$$

$$=\int_0^1\left(-x+1+x^2-x-\frac{x^2-2x+1}{2}\right)dx=\int_0^1\left(\frac{x^2}{2}-x+\frac{1}{2}\right)dx$$

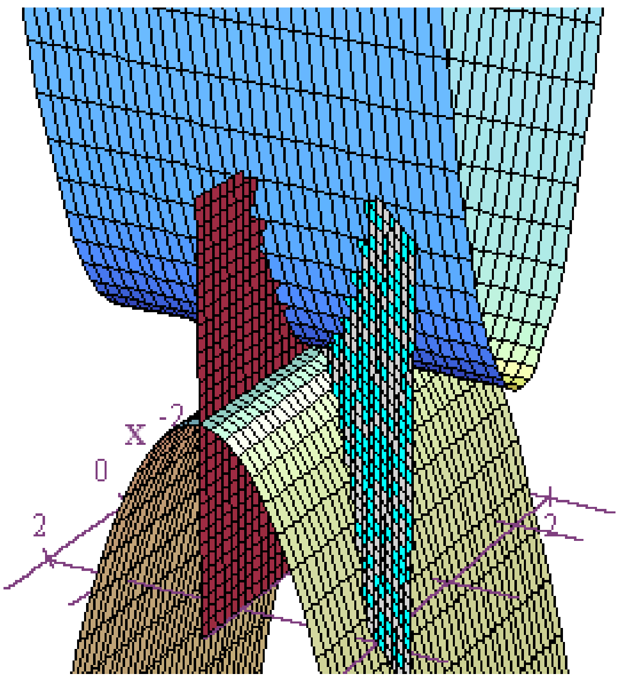
$$=\left(\frac{x^3}{6}-\frac{x^2}{2}+\frac{x}{2}\right)\Bigg|_0^1=\frac{1}{6}$$

Any questions?

Our last example on triple integrals involves integrating a function over a solid volume.

Example 5: Let V be the solid region between the graphs of $z=-y^2$ and $z=x^2$ where $0\le x\le 1$ and $0\le y\le x$. Evaluate $\iiint_V (x+1)\,dV$.

Solution: The solid bounded by a variety of surfaces is not that easy to draw, but here's part of it.



Fortunately, it's still pretty easy to see how to describe our solid region in terms of inequalities.

$$0 \le x \le 1$$
$$0 \le y \le x$$
$$-y^2 \le z \le x^2$$

Thus,

$$\text{Volume } = \iiint_V dV = \int_0^1 \int_0^x \int_{-y^2}^{x^2} dz dy dx = \int_0^1 \int_0^x \left(x^2 + y^2\right) dy dx$$
$$= \int_0^1 \left(x^2 y + \frac{y^3}{3} \right)\Bigg|_0^x dx = \int_0^1 \frac{4x^3}{3} dx = \frac{x^4}{3}\Bigg|_0^1 = \frac{1}{3}$$

Pretty simple!

Quite often what is a difficult integral in *xyz*-coordinates can be a very simple integral in other coordinate systems. At this point, the only other coordinate systems we are familiar with are polar, cylindrical, and spherical coordinates. We'll begin with double integrals in polar coordinates. It's often advantageous to switch to this coordinate system if the region *R* we are integrating over is circular or otherwise easy to describe in the polar coordinate system. Also, recall the following important relationships between polar coordinates and *x* and *y*.

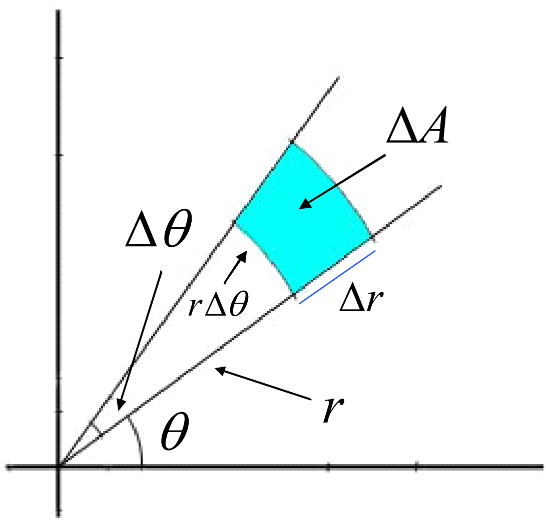
$$x = r \cdot \cos\theta$$
$$y = r \cdot \sin\theta$$
$$x^2 + y^2 = r^2$$

Recall also that when dealing with rectangular coordinates, we were able to write a double integral as an iterated integral with respect to *y* and *x* in the way we did because an element of area in our region R was equal to the product of a change in y times a change in x.

$$\Delta A = \Delta y \Delta x \Rightarrow dA = dydx$$

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dydx$$

Consequently, the question is if we are going to write $\iint_R f(x,y)\,dA$ as an iterated integral with respect to r and θ, then how do we express an element of area in terms of these input variables? The diagram below gives us the answer.



$$\Delta A \approx r\Delta\theta \cdot \Delta r = r\Delta r\Delta\theta$$
$$dA = rdrd\theta$$

In polar coordinates, our elements of area are portions of "pizza slices." Consequently, they aren't the usual rectangles that we have when dealing with *xy*-

coordinates. Nonetheless, if our changes in r and θ are small, then our regions will approximate rectangles.

If we look at the shaded region above, we see that one side of it corresponds to the change in radius, Δr, and the other side corresponds to the length of a circular arc where the radius is r and the angle is represented by $\Delta\theta$. In this case, the arc length is $r\Delta\theta$. And since our element of area is approximately rectangular when the changes in our variables are small, we can unequivocally state that,

$$\Delta A \approx \Delta r \cdot r\Delta\theta = r\Delta r\Delta\theta$$

This tells us that the corresponding formula for differentials is,

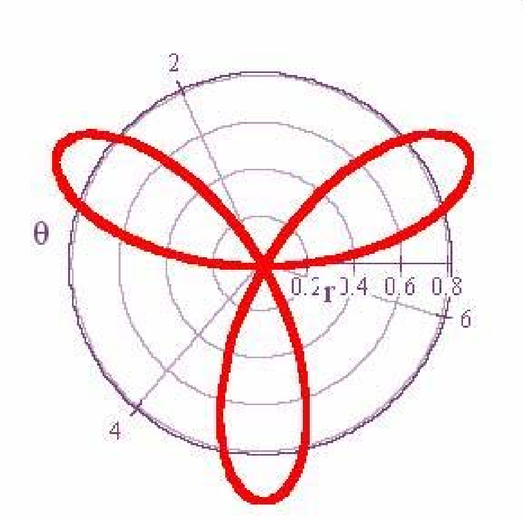
$$dA = r\,drd\theta$$

Hence, if we make this substitution for dA in our double integral, then we get

$$\iint_R f(x,y)\,dA = \int_e^f \int_g^h f(r\cos\theta, r\sin\theta)\,rdrd\theta$$

In other words, to write our double integral as an iterated integral in polar coordinates, we have to replace dA by $r\,drd\theta$, the variable x by $r\cos\theta$, and the variable y by $r\sin\theta$. Additionally, our limits of integration have to be changed to reflect the corresponding intervals for r and θ. Let's look at a few examples.

Example 6: Find the area of one petal of the three petal rose that is the graph of $r = \sin(3\theta)$.



Solution: Let $\begin{cases} z = 1 \\ 0 \le \theta \le \dfrac{\pi}{3} \\ 0 \le r \le \sin(3\theta) \end{cases}$. Then,

$$\text{Area } = \iint_R dA = \int_0^{\pi/3}\int_0^{\sin(3\theta)} r\,dr\,d\theta = \int_0^{\pi/3} \left.\frac{r^2}{2}\right|_0^{\sin(3\theta)} d\theta = \int_0^{\pi/3} \frac{\sin^2(3\theta)}{2}\,d\theta$$

$$= \frac{1}{6}\int_0^{\pi} \sin^2 u\,du = \frac{1}{6}\int_0^{\pi} \frac{1-\cos 2u}{2}\,du = \left.\frac{1}{6}\left(\frac{u}{2} - \frac{\sin 2u}{4}\right)\right|_0^{\pi} = \frac{\pi}{12}. \quad \blacksquare$$

As you can see, one of the tricks is to simply describe your region properly in terms of polar coordinates. Once you've accomplished that, the other substitutions are usually pretty easy. Now let's look at another example where our polar limits of integration are already given to us.

Example 7: Find the integral of $z = \dfrac{1}{(x^2+y^2)^{3/2}}$ on the region R corresponding to

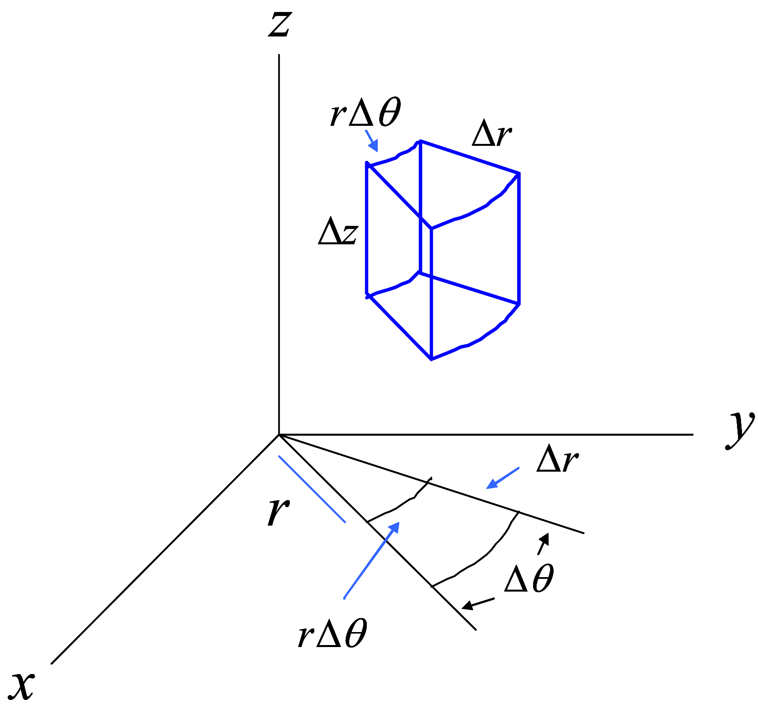
$0 \le \theta \le \dfrac{\pi}{4}$ and $1 \le r \le 2$.

Solution: Since our integrand involves the expression x^2+y^2, we can immediately replace this by r^2. Thus,

$$\iint_R \frac{1}{(x^2+y^2)^{3/2}}\,dA = \int_0^{\pi/4}\int_1^2 \frac{1}{(r^2)^{3/2}}\,r\,drd\theta = \int_0^{\pi/4}\int_1^2 r^{-2}\,drd\theta$$

$$= \int_0^{\pi/4} -\frac{1}{r}\bigg|_1^2\,d\theta = \int_0^{\pi/4}\frac{1}{2}d\theta = \frac{\theta}{2}\bigg|_0^{\pi/4} = \frac{\pi}{8}. \quad \blacksquare$$

If we are using a triple integral to find a volume and if the corresponding region in the *xy*-plane is easily described by polar coordinates, then it might be a good idea to express the triple integral in cylindrical coordinates. Recall that in cylindrical coordinates, the first two coordinates are polar and the third coordinate is still z as in the usual *xyz*-rectangular system. Now to properly convert to cylindrical coordinates, the trick is going to be to figure out how an element of volume would be represented in this coordinate system. The following diagram will guide us in how to do this.

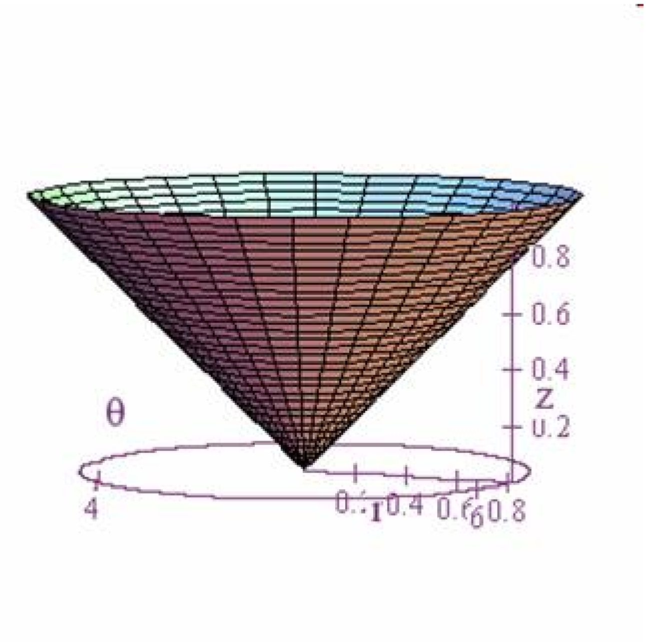


$$\Delta V \approx r\Delta\theta \cdot \Delta z \cdot \Delta r = r\Delta z\Delta r\Delta\theta$$

$$dV = r\,dzdrd\theta$$

Again, when the changes in our variables are small, the element of volume can be approximated by a cube. From the diagram above we see that we can take the height of this cube to be Δz, and the sides of the base to be Δr and $r\Delta\theta$. That results in the approximation we see above that $\Delta V \approx r\Delta\theta\Delta z\Delta r = r\Delta z\Delta r\Delta\theta$. Or as we write in the differential version, $dV = r\,dzdrd\theta$. Now let's take an example or two.

Example 8: Find the volume of the region above the *xy*-plane and beneath the cone defined by $z = \sqrt{x^2 + y^2} = \sqrt{r^2} = r$, $0 \le r \le 1$, and $0 \le \theta \le 2\pi$.



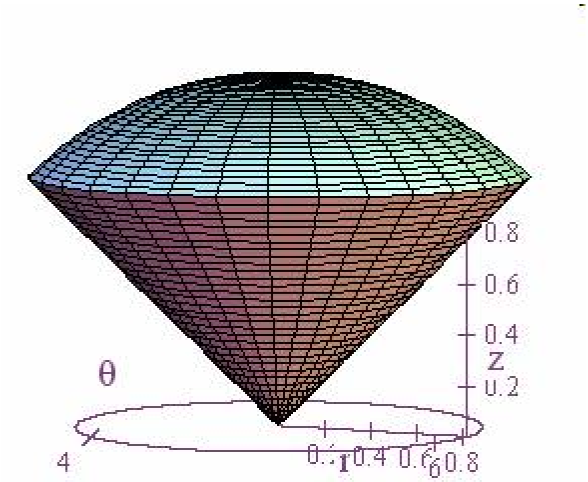
Solution: In this case, the range for *z* will be $0 \le z \le r$. Thus,

$$\text{Volume } = \iiint_V dV = \int_0^{2\pi}\int_0^1\int_0^r r\,dz dr d\theta = \int_0^{2\pi}\int_0^1 rz\Big|_0^r \,dr d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^2\,dr d\theta = \int_0^{2\pi} \frac{r^3}{3}\Big|_0^1 d\theta = \int_0^{2\pi}\frac{1}{3}d\theta = \frac{\theta}{3}\Big|_0^{2\pi} = \frac{2\pi}{3}. \quad \blacksquare$$

One of my favorite uses of polar coordinates is to find the volume of a filled ice cream cone such as we do in the next example.

Example 9: Find the volume of the ice cream cone defined by $r \le z \le \sqrt{2-r^2}$, $0 \le r \le 1$, and $0 \le \theta \le 2\pi$.



Solution: There are a couple of things we should probably point out before getting started. First, the ranges for the polar coordinates, $0 \le r \le 1$ and $0 \le \theta \le 2\pi$, describe the unit circle in the *xy*-plane, the circle of radius 1 with center at the origin. Second, $z = r$ is going to give us the same cone that we looked at in the previous example. The ice cream that is put in the cone, however, is what we claim is described by $z = \sqrt{2-r^2}$. To see why this is so, recall that $r^2 = x^2 + y^2$. Hence,

$z = \sqrt{2-r^2} = \sqrt{2-(x^2+y^2)} \Rightarrow z^2 = 2-(x^2+y^2) \Rightarrow x^2+y^2+z^2 = 2$. This last equation is nothing more than the equation for a sphere of radius $\sqrt{2}$ with center at the origin. However, the restrictions in the *xy*-plane that $0 \le r \le 1$ and $0 \le \theta \le 2\pi$ result in us getting only that portion of the sphere that tops off the ice cream cone. Now let's find the volume.

$$\text{Volume } = \iiint_V dV = \int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r\,dzdrd\theta = \int_0^{2\pi}\int_0^1 rz\Big|_r^{\sqrt{2-r^2}} drd\theta$$

$$= \int_0^{2\pi}\int_0^1\left(r\sqrt{2-r^2}-r^2\right)drd\theta = \int_0^{2\pi}\left(\frac{-(2-r^2)^{3/2}}{3}-\frac{r^3}{3}\right)\Bigg|_0^1 d\theta$$

$$= \int_0^{2\pi}\frac{(2^{3/2}-2)}{3}d\theta = \frac{\theta(2^{3/2}-2)}{3}\Bigg|_0^{2\pi} = \frac{2\pi(2^{3/2}-2)}{3} = \frac{4\pi}{3}\left(\sqrt{2}-1\right). \quad \blacksquare$$

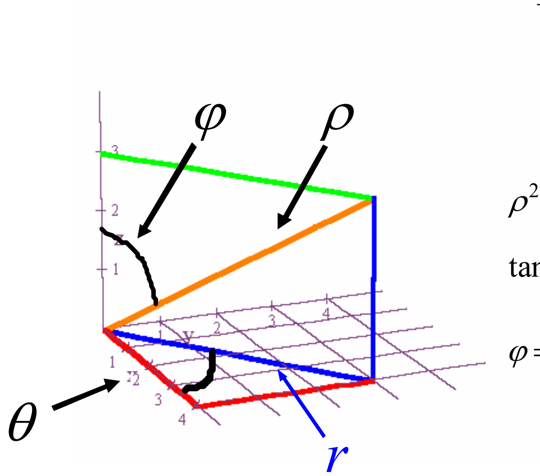
I love combining math with food! Also, note that if we were trying to find this same volume using rectangular coordinates, then our integral would look like this.

$$\text{Volume } = \iiint_V dV = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dzdydx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\left(\sqrt{2-x^2-y^2}-\sqrt{x^2+y^2}\right)dydx$$

Trust me. This integral is not a piece of cake in rectangular coordinates. Changing to cylindrical coordinates makes the problem much easier.

We'll soon redo the same problem in spherical coordinates. First, though, recall some of the basic formulas for converting between rectangular and spherical coordinates.



$$(\rho,\theta,\varphi)$$

$$\rho^2 = x^2 + y^2 + z^2$$

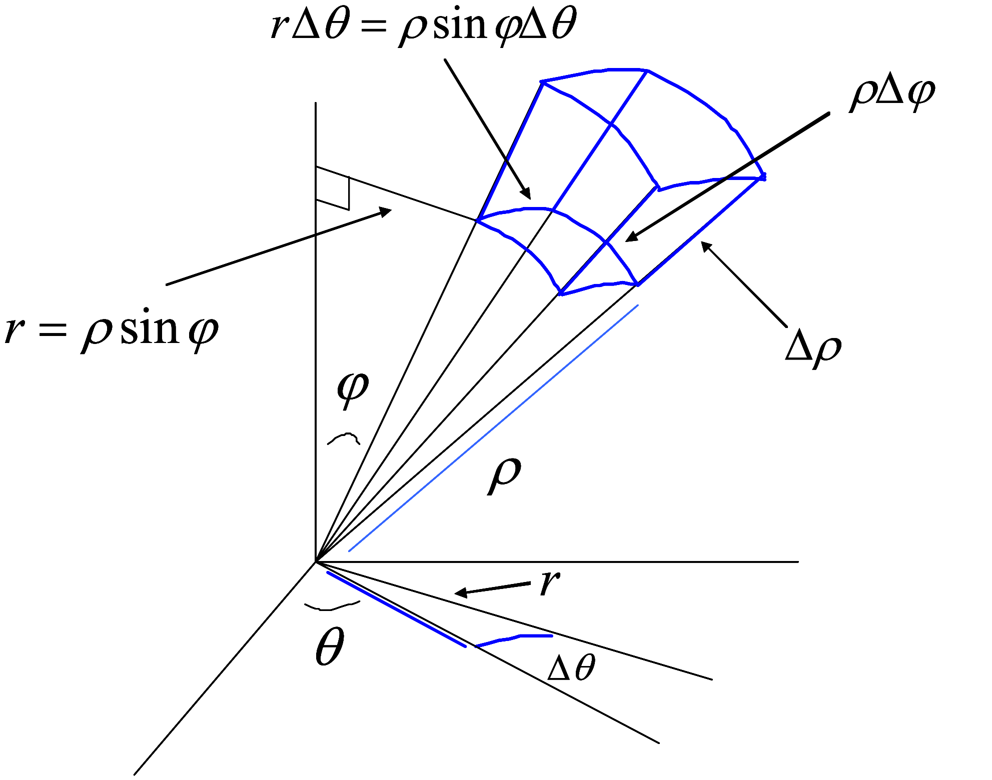
$$\tan\theta = \frac{y}{x}$$

$$\varphi = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$x = r\cos(\theta) = \rho\sin(\varphi)\cos(\theta)$$
$$y = r\sin(\theta) = \rho\sin(\varphi)\sin(\theta)$$
$$z = \rho\cos(\varphi)$$

$$0 \le \rho < \infty$$
$$0 \le \varphi \le \pi$$
$$0 \le \theta < 2\pi$$

Since we've gone over this diagram before, we'll leave it to you to do the math this time. Instead, we'll focus now on how to describe an element of volume in spherical coordinates. As you might expect, this element will be a piece of a sphere, and as the changes in our variables get smaller, this piece will approximate a cube.



$$\Delta V \approx \Delta\rho\cdot\rho\Delta\varphi\cdot\rho\sin\varphi\Delta\theta = \rho^2\sin\varphi\,\Delta\rho\Delta\varphi\Delta\theta$$

From this diagram we can see that our element of volume is,

$$\Delta V \approx \Delta\rho\cdot\rho\Delta\varphi\cdot r\Delta\theta = \Delta\rho\cdot\rho\Delta\varphi\cdot\rho\sin\varphi\Delta\theta = \rho^2\sin\varphi\,\Delta\rho\Delta\varphi\Delta\theta$$

In differential form this becomes,

$$dV = \rho^2\sin\varphi\;\; d\rho\; d\varphi\; d\theta$$

Thus, in spherical coordinates our triple integral for volume becomes the following iterated integral,

$$\text{Volume } = \iiint_V dV = \int_a^b\int_c^d\int_e^f \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

Using spherical coordinates, we can now easily show that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

Example 10: Find the volume of a sphere of radius r with center at the origin.

Solution: We simply set up a triple integral with the following limits and integrate.

$$0 \le \rho \le r$$
$$0 \le \varphi \le \pi$$
$$0 \le \theta \le 2\pi$$

$$\text{Volume } = \iiint_V dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \int_0^{2\pi}\int_0^{\pi} \left.\frac{\rho^3 \sin\varphi}{3}\right|_0^r d\varphi d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi} \frac{r^3 \sin\varphi}{3} d\varphi d\theta = \int_0^{2\pi} \left. -\frac{r^3 \cos\varphi}{3}\right|_0^{\pi} d\theta = \int_0^{2\pi} \frac{2r^3}{3} d\theta = \left.\frac{2r^3\theta}{3}\right|_0^{2\pi} = \frac{4}{3}\pi r^3$$

Now we'll use spherical coordinates to tackle the ice cream problem!

Example 11: Find the volume of the ice cream cone defined by $0 \le \theta \le 2\pi$, $0 \le \varphi \le \frac{\pi}{4}$, and $0 \le \rho \le \sqrt{2}$.

<u>Solution:</u> First, notice how incredibly easy it is to describe this solid region in spherical coordinates. Our angle θ in the *xy*-plane goes full circle from 0 to 2π, our angle φ with the positive *z*-axis goes from 0 to $\frac{\pi}{4}$, and the sphere radius goes from 0 to $\sqrt{2}$. It's as easy as π! And now that we understand our limits of integration, we can find the volume.

$$\text{Volume } = \iiint_V dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^3}{3}\sin\varphi \Big|_0^{\sqrt{2}} d\varphi d\theta = \int_0^{2\pi}\int_0^{\pi/4} \frac{2^{3/2}}{3}\sin\varphi \, d\varphi d\theta$$

$$= \int_0^{2\pi} \frac{2^{3/2}}{3}(-\cos\varphi)\Big|_0^{\pi/4} d\theta = \int_0^{2\pi} \frac{2^{3/2}}{3}\left(\frac{-1}{\sqrt{2}}+1\right)d\theta$$

$$= \frac{\theta\cdot 2^{3/2}}{3}\left(\frac{-1}{\sqrt{2}}+1\right)\Big|_0^{2\pi} = \frac{2\pi}{3}\cdot 2^{3/2}\left(\frac{-1}{\sqrt{2}}+1\right) = \frac{4\pi}{3}\cdot\sqrt{2}\left(\frac{-1}{\sqrt{2}}+1\right)$$

$$= \frac{4\pi}{3}\left(\sqrt{2}-1\right). \quad \blacksquare$$

Here's another example that's tailor made for spherical coordinates.

<u>Example 12:</u> Find $\iiint_V z^2 \, dV$ on the region between the spheres with radii $\rho = 1$ and $\rho = 2$.

Solution: Notice in this problem that we have $\begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \\ 1 \le \rho \le 2 \\ z = \rho\cos\varphi \Rightarrow z^2 = \rho^2\cos^2\varphi \end{cases}$.

Hence,

$$\iiint_V z^2\,dV = \int_0^{2\pi}\int_0^{\pi}\int_1^{2} \rho^2\cos^2\varphi\,\rho^2\sin\varphi\;d\rho d\varphi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\left(\frac{\rho^5}{5}\cos^2\varphi\sin\varphi\right)\Bigg|_1^2 d\varphi d\theta = \int_0^{2\pi}\int_0^{\pi}\frac{31}{5}\cos^2\varphi\sin\varphi\,d\varphi d\theta$$

$$= \int_0^{2\pi}\frac{31}{5}\frac{\left(-\cos^3\varphi\right)}{3}\Bigg|_0^{\pi} d\theta = \int_0^{2\pi}\frac{62}{15}d\theta = \frac{62\theta}{15}\Bigg|_0^{2\pi}$$

$$= \frac{62\cdot 2\pi}{15} - \frac{62\cdot 0}{15} = \frac{124\pi}{15}. \quad \blacksquare$$

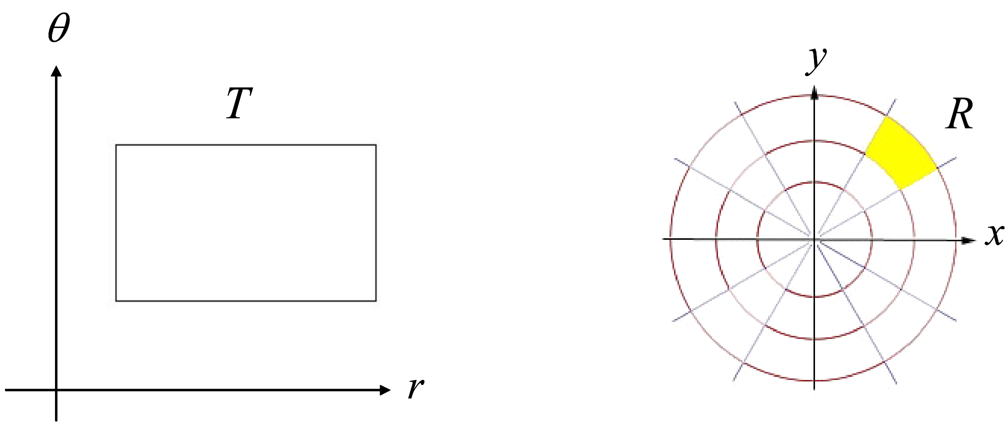
What could be simpler!

The above examples certainly illustrate how changing to a different coordinate system can often make an integration easier to do. However, our limitation has been that we only had a few different coordinate systems to convert to, polar, cylindrical, or spherical. Nonetheless, these examples raise the question of whether it is possible to find a general formula for changing from rectangular coordinates to any coordinate system whatsoever. This last part of our chapter is devoted to accomplishing this task, and to help us develop a more general procedure, we'll take another look at what goes on when we change from rectangular to polar coordinates.

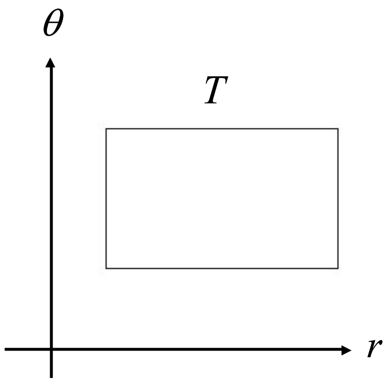
When we convert a double integral from rectangular to polar coordinates, recall the changes that must be made to x, y, and dA.

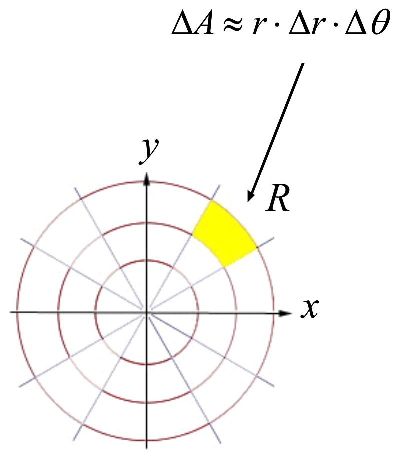
$$x = x(r,\theta) = r\cos\theta$$
$$y = y(r,\theta) = r\sin\theta$$
$$dA = r\,drd\theta$$

In the polar coordinate system, an element of area is generally a rectangle corresponding to a range of values for r and θ.



However, as you can see above, in the xy-coordinate system, this rectangle takes on a different shape, and the formula for an element of area changes.





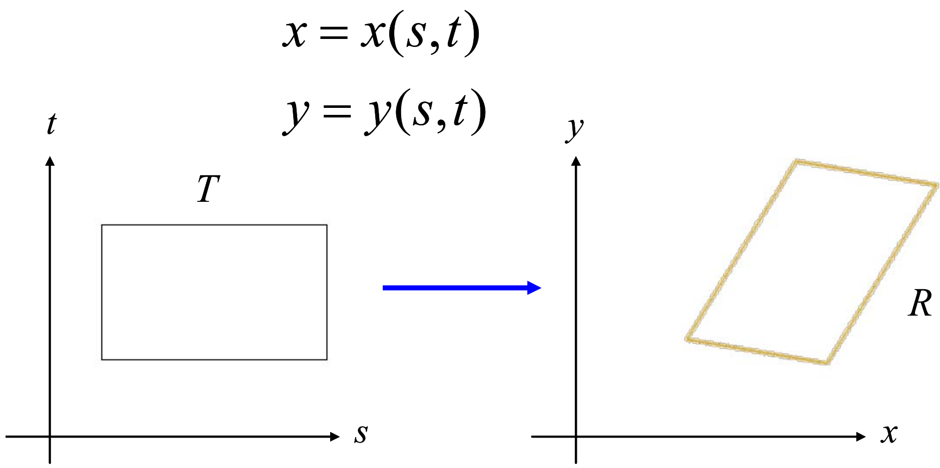
And of course, this leads to the following formula for double integrals in polar coordinates.

$$\iint_R f(x,y)\,dA = \iint_T f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

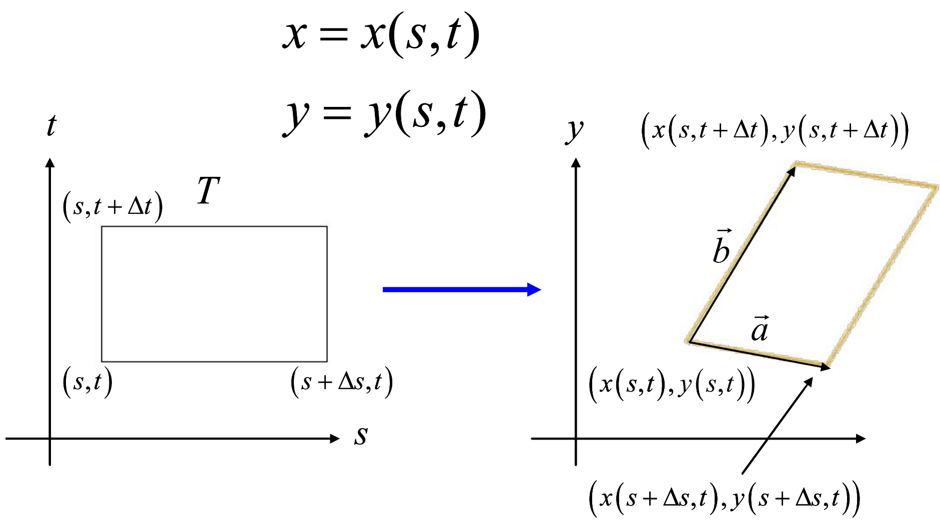
Using the polar coordinate example as a model, we now want to develop a general method for finding change of variable formulas such as the polar coordinate one. Thus, suppose we have a rectangle in an *st*-coordinate system and a pair of functions that converts (s,t) coordinates into (x,y) coordinates.

$$x = x(s,t)$$
$$y = y(s,t)$$

Suppose also that these functions are differentiable and that the transformation from the *st*-coordinates to *xy*-coordinates is one-to-one. Then because of differentiability, local linearity will be present and a small rectangle in the *st*-coordinate system will be mapped onto approximately a parallelogram in the *xy*-coordinate system.

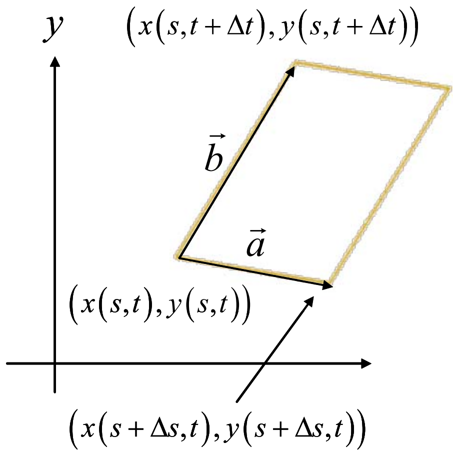


If we add some coordinates, then it looks like this.



Since an element of area in our *xy*-coordinate system is represented by a parallelogram, the area of this parallelogram is equal to $\left\|\vec{a}\times\vec{b}\right\|$ where $\vec{a}$ and $\vec{b}$ are vectors that correspond to the two sides of the parallelogram.

$$Area = \left\|\vec{a}\times\vec{b}\right\|$$



However, notice that,

$$\begin{aligned}
\vec{a} &= \left(x(s+\Delta s,t)-x(s,t)\right)\hat{i}+\left(y(s+\Delta s,t)-y(s,t)\right)\hat{j} \\
&\approx \frac{\partial x}{\partial s}\Delta s\,\hat{i}+\frac{\partial y}{\partial s}\Delta s\,\hat{j}
\end{aligned}$$

$$\begin{aligned}
\vec{b} &= \left(x(s,t+\Delta t)-x(s,t)\right)\hat{i}+\left(y(s,t+\Delta t)-y(s,t)\right)\hat{j} \\
&\approx \frac{\partial x}{\partial t}\Delta t\,\hat{i}+\frac{\partial y}{\partial t}\Delta t\,\hat{j}
\end{aligned}$$

Hence,

$$\vec{a}\times\vec{b}\approx\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial x}{\partial s}\Delta s & \frac{\partial y}{\partial s}\Delta s & 0\\ \frac{\partial x}{\partial t}\Delta t & \frac{\partial y}{\partial t}\Delta t & 0\end{vmatrix}=\left(\frac{\partial x}{\partial s}\Delta s\cdot\frac{\partial y}{\partial t}\Delta t-\frac{\partial y}{\partial s}\Delta s\cdot\frac{\partial x}{\partial t}\Delta t\right)\hat{k}$$

Furthermore,

$$\left\|\vec{a}\times\vec{b}\right\|\approx\left|\frac{\partial x}{\partial s}\Delta s\cdot\frac{\partial y}{\partial t}\Delta t-\frac{\partial y}{\partial s}\Delta s\cdot\frac{\partial x}{\partial t}\Delta t\right|=\left|\frac{\partial x}{\partial s}\frac{\partial y}{\partial t}-\frac{\partial y}{\partial s}\frac{\partial x}{\partial t}\right|\Delta s\Delta t$$

The expression inside the last absolute value sign is called the *Jacobian*, and it is usually written as,

$$\frac{\partial(x,y)}{\partial(s,t)}=\begin{vmatrix}\frac{\partial x}{\partial s} & \frac{\partial x}{\partial t}\\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t}\end{vmatrix}=\frac{\partial x}{\partial s}\frac{\partial y}{\partial t}-\frac{\partial y}{\partial s}\frac{\partial x}{\partial t}$$

Consequently, the area of our parallelogram is equal to the absolute value of the *Jacobian* times the change in *s* and the change in *t*.

$$Area=\Delta A=\left\|\vec{a}\times\vec{b}\right\|\approx\left|\frac{\partial x}{\partial s}\frac{\partial y}{\partial t}-\frac{\partial y}{\partial s}\frac{\partial x}{\partial t}\right|\Delta s\Delta t=\left|\frac{\partial\left(x,y\right)}{\partial\left(s,t\right)}\right|\Delta s\Delta t$$

And this tells us exactly what to substitute for dA in our integral formula,

$$dA = \left|\frac{\partial(x,y)}{\partial(s,t)}\right| dsdt$$

$$\iint_R f(x,y)\,dA = \lim_{\Delta A\to 0}\sum f(x,y)\cdot\Delta A$$
$$= \lim_{\Delta s,\Delta t\to 0}\sum f(x(s,t),y(s,t))\cdot\left|\frac{\partial(x,y)}{\partial(s,t)}\right|\Delta s\Delta t$$
$$= \iint_T f(x(s,t),y(s,t))\cdot\left|\frac{\partial(x,y)}{\partial(s,t)}\right| dsdt$$

Now let's verify that this formula works for transformations to polar coordinates.

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$
$$= r\cos^2\theta + r\sin^2\theta = r\left(\cos^2\theta + \sin^2\theta\right) = r$$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)} = \right| = |r| = r$$

$$dA = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta = r\,drd\theta$$

Bingo!

$$\iint_R f(x,y)\,dA = \iint_T f(r\cos\theta, r\sin\theta)\cdot\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta$$
$$= \iint_T f(r\cos\theta, r\sin\theta)\cdot r\,drd\theta$$

Now let's try to figure out how to transform an ellipse into a circle. Suppose we start with the equation for an ellipse below, and we'll assume that both *a* and *b* are positive.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

If we set $\begin{cases} x = a \cdot s \\ y = b \cdot t \end{cases}$, then

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{(a \cdot s)^2}{a^2}+\frac{(b \cdot t)^2}{b^2}=1 \Rightarrow s^2+t^2=1$$

Hence, the *Jacobian* of this transformation is,

$$\frac{\partial(x,y)}{\partial(s,t)}=\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix}=\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}=ab$$

And the absolute value of the *Jacobian* is,

$$\left|\frac{\partial(x,y)}{\partial(s,t)}\right|=ab$$

Using this transformation, we can now easily find the area of the ellipse.

$$\begin{aligned} Area\,of\;ellipse &= \iint_{ellipse} dA = \iint_{unit\,circle} ab\,dsdt \\ &= ab\left(\iint_{unit\,circle} dsdt\right) = ab \cdot \pi = \pi ab \end{aligned}$$

It's a piece of lettuce! Nothing to it!

Everything we've done involving changing coordinate systems carries over to higher dimensions, too. In fact, if we have transformation involving three variables, then our *Jacobian* looks like this,

$$\frac{\partial(x,y,z)}{\partial(s,t,u)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} & \frac{\partial z}{\partial u} \end{vmatrix}$$

Here's an example of how find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by first using a change of variables to transform the ellipsoid into a sphere. Again, we will assume that *a*, *b*, and *c* are positive.

Let $\begin{cases} x = a \cdot s \\ y = b \cdot t \\ z = c \cdot u \end{cases}$. Then this will transform our ellipsoid into a sphere of radius 1.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \Rightarrow \frac{a^2s^2}{a^2}+\frac{b^2t^2}{b^2}+\frac{c^2u^2}{c^2}=1 \Rightarrow s^2+t^2+u^2=1$$

Furthermore, recall that the volume of a sphere of radius *r* is given by the formula

$V = \frac{4}{3}\pi r^3$. Hence, in this case with $r = 1$, the volume is just $\frac{4\pi}{3}$.

Additionally, the *Jacobian* of this transformation is,

$$\frac{\partial(x,y,z)}{\partial(s,t,u)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} & \frac{\partial z}{\partial u} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

Thus, the volume of our ellipsoid is,

$$Volume = \iiint_R dV = \iiint_T \left| \frac{\partial(x,y,z)}{\partial(s,t,u)} \right| dsdtdu = \iiint_T abc\, dsdtdu$$
$$= abc \iiint_T dsdtdu = abc \cdot \frac{4\pi}{3} = \frac{4}{3}\pi abc$$

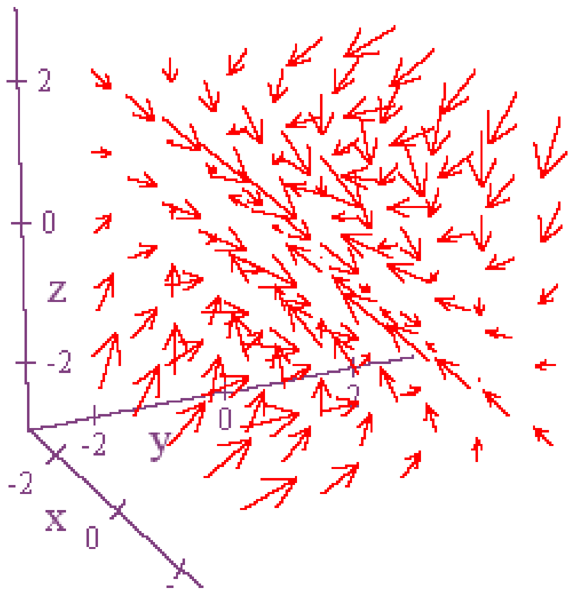
Any questions? 

CHAPTER 9

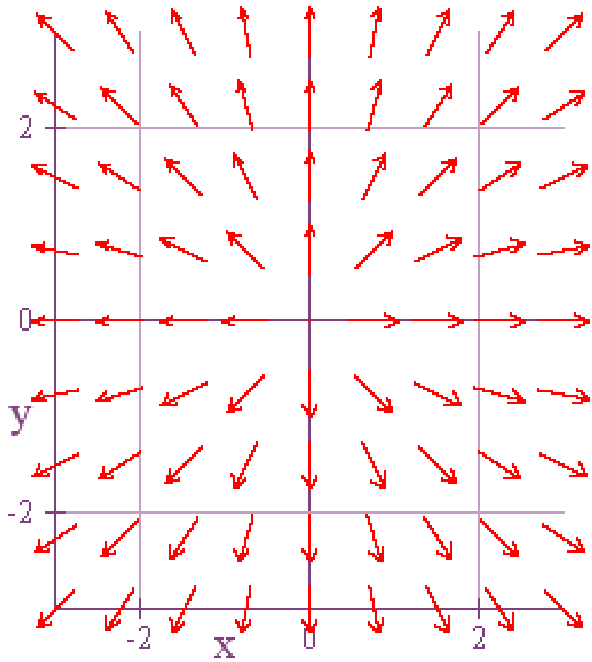
VECTOR FIELDS

I'll always remember quite vividly how confused I was when I heard that vectors had magnitude and direction but didn't exist at any particular location in space. This was confusing to me because the common interpretation of a vector was that of a force, and forces always acted at particular locations. For example, a river might be raging at one point and the water might be calm at another, and there is no way that the force at the former point can be arbitrarily moved to a different location. It was only much later that I understood that mathematicians didn't force (no pun intended!) vectors to have a location in space because they wanted to be able to move them around like legos or tinker toys so that they could easily be added together by placing the initial point of one vector on top of the terminal point of another. Furthermore, if vectors possessed location, then something like $\vec{v} = 2\hat{i} + 3\hat{j}$ would have to represent a vector that started at one and only one point, and it would, consequently, be very difficult to define vector arithmetic. Nevertheless, the concept of vectors that are tied to points does result in an important model of the real world, and such a model is what we are going to call a vector field.

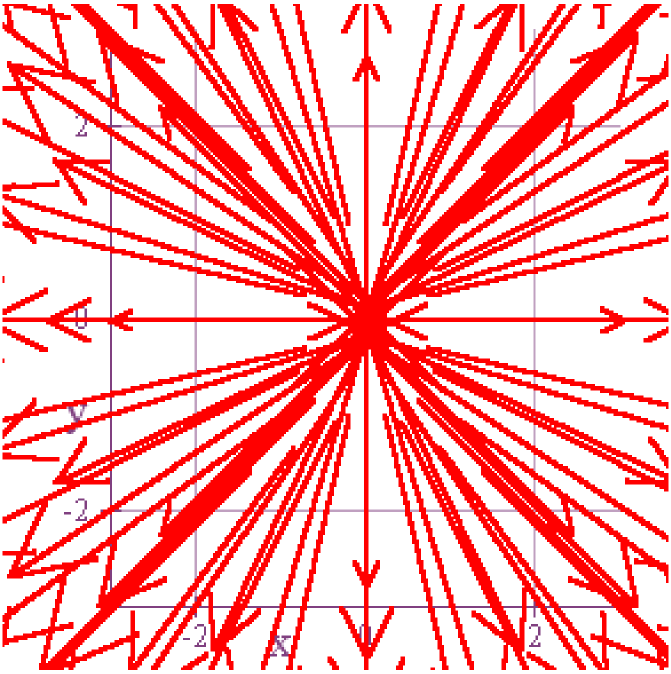
Technically, we'll say that a vector field is a function that assigns a vector to a point in *n*-dimensional space. For example, if to each point (x, y, z) we assign the vector $\vec{F}(x, y, z) = x\hat{i} - y\hat{j} - z\hat{k}$, then a graph of the resulting vector field looks like this.



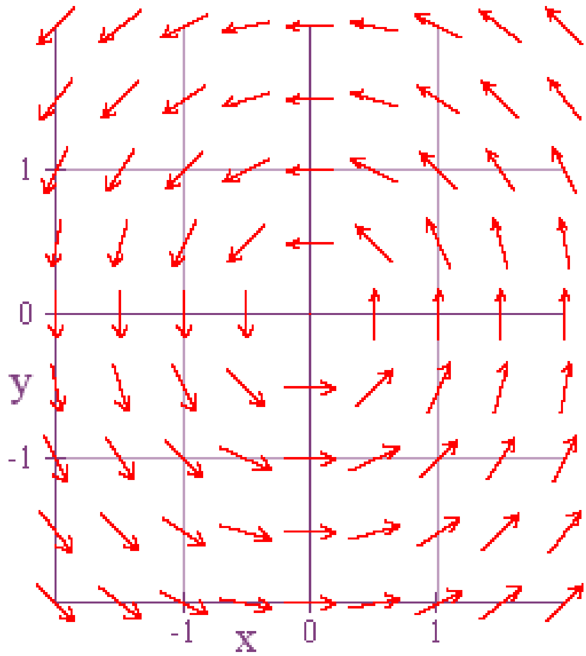
As you can see, however, even using software to draw this thing, it's still a bit of a mess. Thus, we're going to spend most of our time looking at vectors fields in just two dimensions. These will be far easier to analyze and understand, and as a result, they will do a better job of illustrating the concepts we are interested in. Below is a graph of the vector field $\vec{F}(x, y) = x\hat{i} + y\hat{j}$.



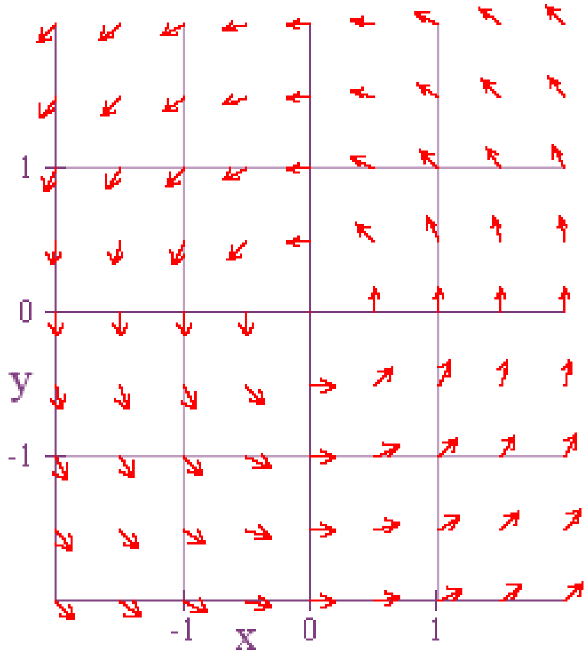
When drawing a vector field, there are a few ways to do it, and it's always a compromise between accuracy and clarity. Technically speaking, the most accurate way to do it would to be to find the vector associated with a point and then draw in that vector by placing the initial point of the vector at our designated point (x, y). In practice, though, the resulting graph is sometimes easier to understand if we draw it so that the midpoint of the vector is at (x, y). Additionally, if we draw each vector with the correct magnitude shown, then things can get messy very quickly. For instance, here is the vector field we just looked at with the true size of each vector shown. As you can see, the result is much harder to understand.



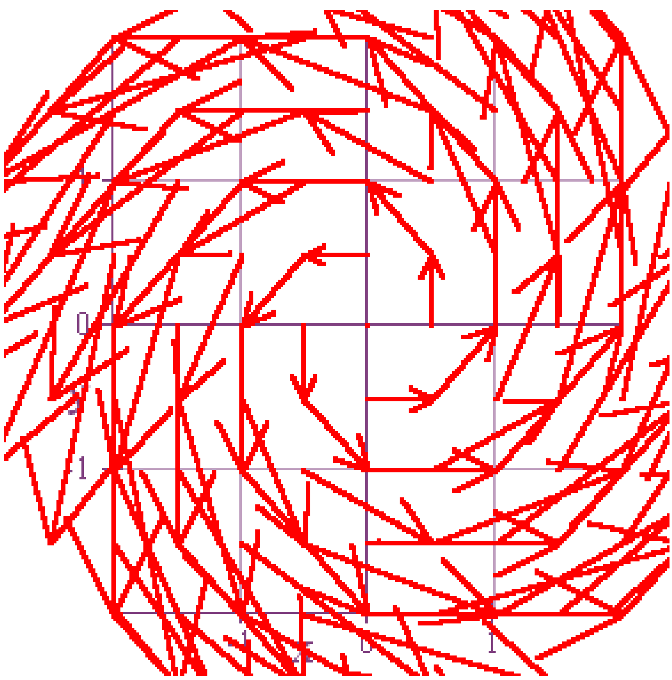
Thus, for the sake of clarity, it's sometimes best to make each vector in your drawing the same length. In practice, this is generally not overly restrictive because we are often interested, primarily, in what we might call the flow pattern of the vectors. In other words, if the vectors represent something like the velocity of a river at each point, then we may simply want to know what would happen if something like a ping pong ball were dropped into the river. Where would the river take it? In the graph of $\vec{F}(x,y) = -y\hat{i} + x\hat{j}$ below, it's easy to understand that the flow of the river would take the ball around in circles.



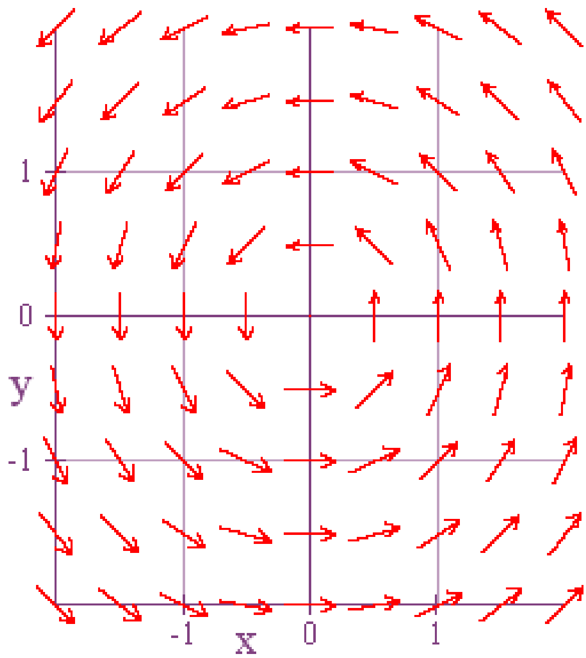
Of course, if we are using software to graph our vector field, then we are always at the mercy of the options available in our program. In the graph above, the point $(1,0)$ gets paired with the vector $\vec{F}(1,0) = -0\hat{i} + 1\hat{j} = \hat{j}$. You can see from the diagram that the vectors are not drawn according to their actual length, and the vector for $(1,0)$ is drawn so that it's midpoint is at the point $(1,0)$. If we adjust things so that the initial point is at $(1,0)$, then our picture looks like this.



And if we also give each vector its true length, then we get this.



A bit of mess, isn't it! The bottom line is that you want to use the graphing options that give you the information you most want, and for me, our original graph is what is easiest to look at in terms of simply understanding the flow created by the vector field.

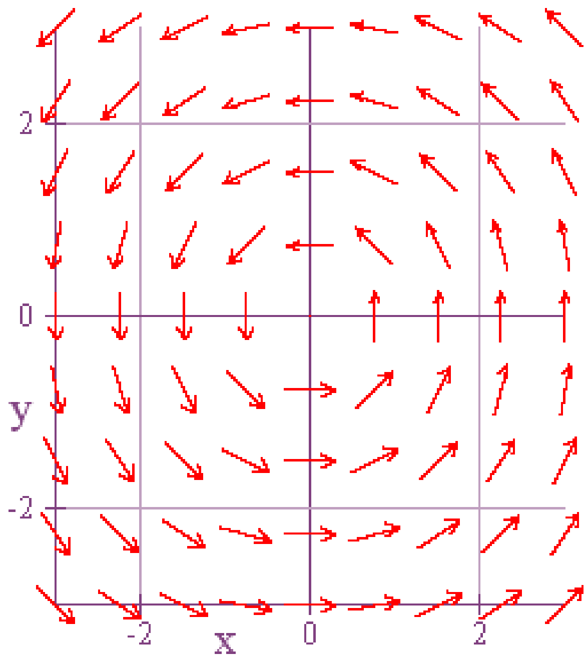


A good thing to do at least once in your life is to take the definition for a vector field and actually create a table of vectors that you might plot. For example, if our vector field is $\vec{F}(x, y) = -y\hat{i} + x\hat{j}$, then a table of values might look like the following.

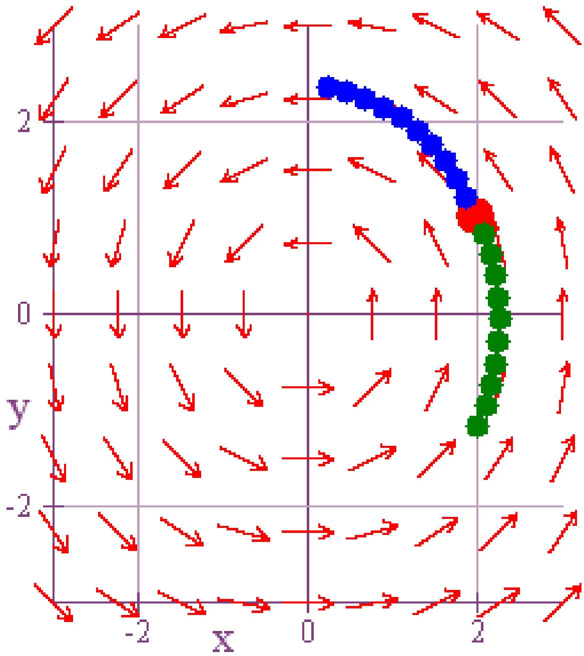
x	y	$\vec{F}(x,y) = -y\hat{i} + x\hat{j}$
0	0	$\vec{0}$
1	0	$\hat{j}$
0	1	$-\hat{i}$
-1	0	$-\hat{j}$
0	-1	$\hat{i}$
2	3	$-3\hat{i} + 2\hat{j}$
-4	1	$-\hat{i} - 4\hat{j}$

Try plotting these vectors by hand, and you'll understand how the diagrams above are created.

If we think of our vector field again as being a flowing river and if we drop a ping pong ball in the river, then the path that it takes is what we'll call a <u>flow line</u>. For the vector field $\vec{F}(x,y) = -y\hat{i} + x\hat{j}$, it's easy to see that all the flow lines are circular.

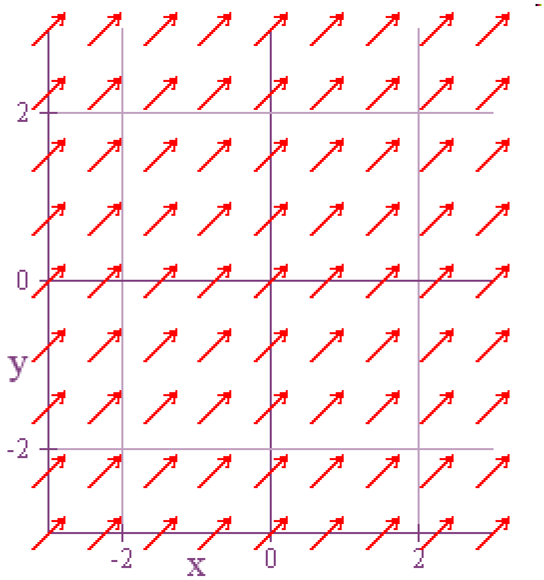


However, we can try and plot a particular flow line by using the following approximation method due to Euler. First, pick a particular point (a,b). This is where we'll drop the ping pong ball. Next, find the vector at that point and multiply it by an increment Δt. In the example I'll show below, I'll set $\Delta t = 0.1$. This means that I'll multiply the vector at (a,b) by a tenth, and then I'll add this vector to my point to get my next point, (a_1,b_1). And then I repeat this procedure several times in order to get several points to plot. To go back in the opposite direction, I just change the sign of my increment, $-\Delta t = -0.1$. As long as your increment is not very large, you'll get a fairly decent plot of points along your flow line. However, as Δt increases in size, your accuracy diminishes. Below is a plot to delineate the flow line when my starting point is $(2,1)$. The starting point is plotted as a big red spot, and the points corresponding to positive increments are plotted in blue while the points resulting from a negative increment are plotted in green. Here's the result.

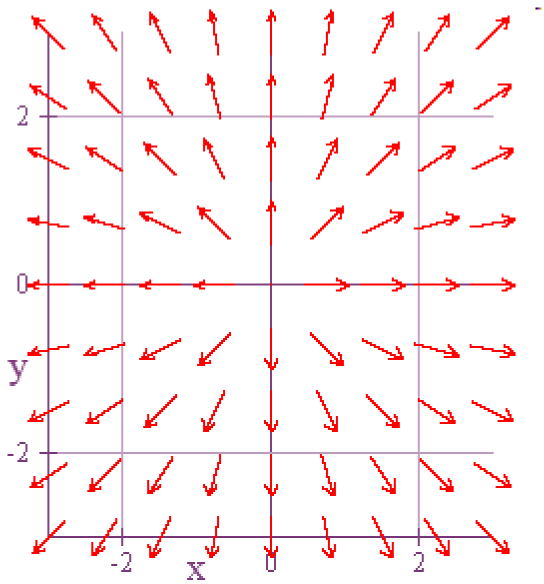


Not bad! Now let's just look at the graphs of a variety of vector fields. As you look at each one, pick a point and try to visualize how the corresponding vector would be drawn at that point. That will help you understand the diagrams with greater ease.

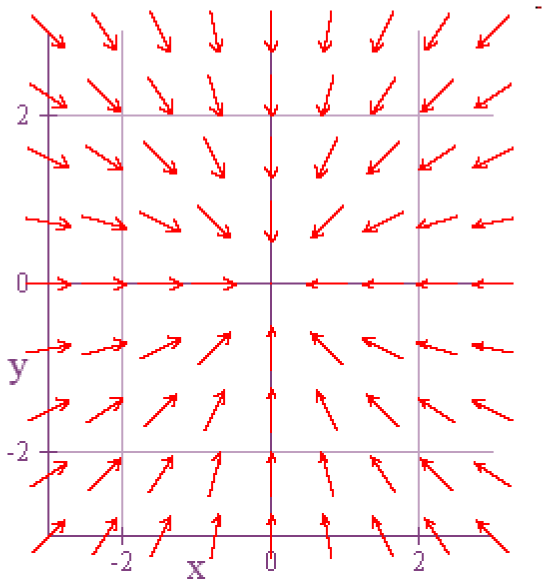
Example 1: $\vec{F}(x,y) = \hat{i} + \hat{j}$



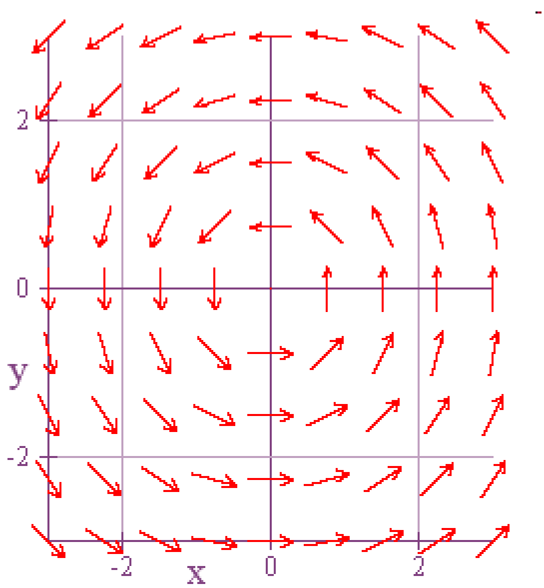
Example 2: $\vec{F}(x,y) = x\hat{i} + y\hat{j}$



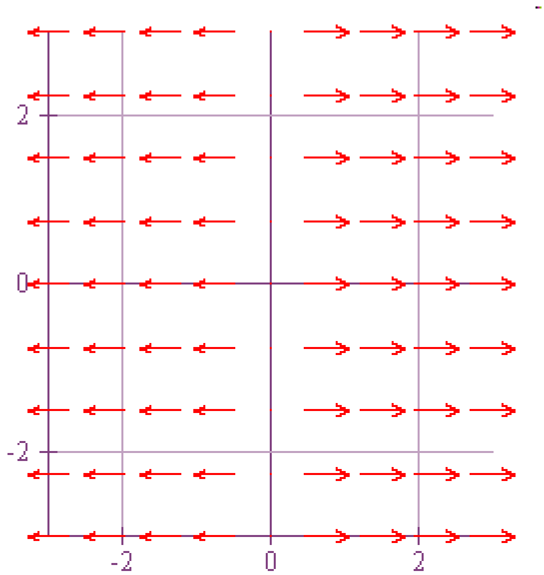
<u>Example 3:</u> $\vec{F}(x,y) = -x\hat{i} - y\hat{j}$



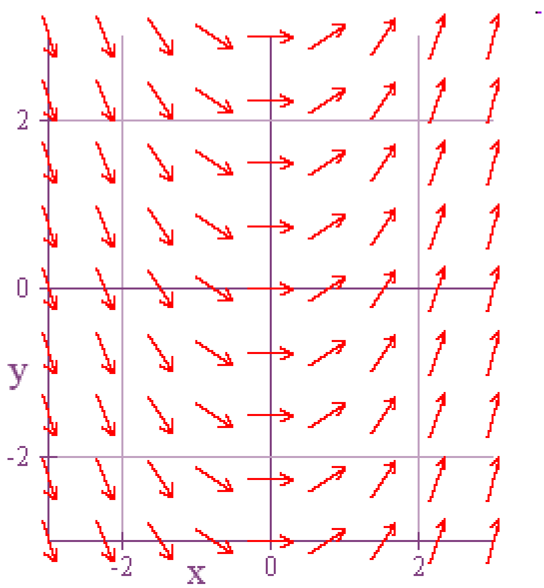
<u>Example 4:</u> $\vec{F}(x,y) = -y\hat{i} + x\hat{j}$



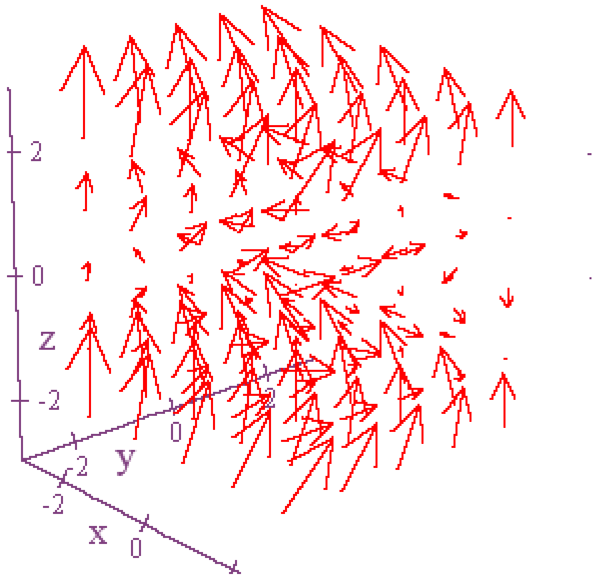
Example 5: $\vec{F}(x,y) = x\hat{i} + 0\,\hat{j} = x\hat{i}$



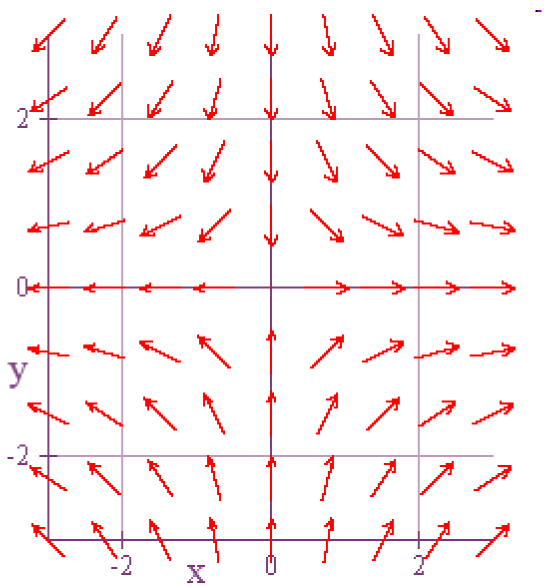
Example 6: $\vec{F}(x,y) = \hat{i} + x\hat{j}$



Example 7: $\vec{F}(x,y,z) = x\hat{i} - y\hat{j} + z^2\hat{k}$

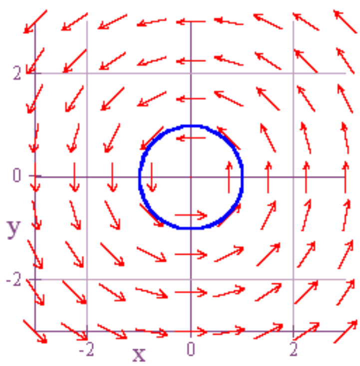


Example 8: $z = f(x,y) = x^2 - y^2,\ \ \nabla f(x,y) = \vec{F}(x,y) = 2x\hat{i} - 2y\hat{j}$

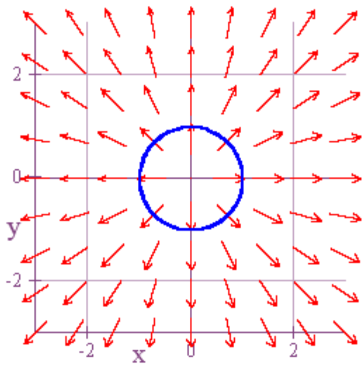


The last example above illustrates something very important. An easy way to get a vector field from a function of several variables is by finding its gradient. The resulting vector field is called a gradient field, and the multivariable function that gives rise to it is called a potential or potential function.

One thing we might notice if we study the vector fields above is that some of them will tend to cause circulation while others will tend to result in flux across a boundary. For example, the vector field below will tend to cause things to circulate counterclockwise around the circle below, but won't tend to move things across the boundary of the circle.



On the other hand, this next vector field will cause a definite outward flux across the circle's boundary, but it won't create any circulation around the circle.



In calculus, there are computations to be done that can measure the tendency of vector field to create either circulation or flux at a particular point. These calculations are called, respectively, the curl and the divergence of the vector field. It won't be apparent until the final chapter of this book why these definitions work, but for now just accept that they do and learn how to do the computations. The definition of the curl of a vector field is as follows.

If $\vec{F} = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$, then

$$\mathit{curl\,of}\ \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ Q & R \end{vmatrix} \hat{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ P & R \end{vmatrix} \hat{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ P & Q \end{vmatrix} \hat{k}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\hat{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\hat{k}$$

If $\vec{F}$ is a 2-dimensional vector field, $\vec{F} = P\hat{i} + Q\hat{j}$, then

$$curl\,of\ \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}$$

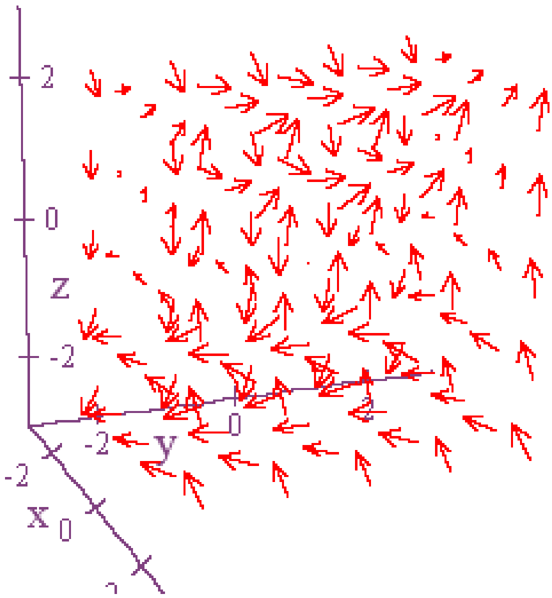
In the definitions above, the cross product notation provides an easy way to remember how to find the curl of a vector field. The symbol ∇ refers to the gradient, but we don't have a function indicated that we should take the gradient of. In this case, the functions come from the components of $\vec{F}$. Thus, if $\vec{F}(x,y,z) = -y^2\,\hat{i} + 2z\,\hat{j} + x^3\,\hat{k}$, then the curl of $\vec{F}$ is,

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & 2z & x^3 \end{vmatrix} = \left(\frac{\partial (x^3)}{\partial y} - \frac{\partial (2z)}{\partial z} \right) \hat{i} - \left(\frac{\partial (x^3)}{\partial x} - \frac{\partial (-y^2)}{\partial z} \right) \hat{j} + \left(\frac{\partial (2z)}{\partial x} - \frac{\partial (-y^2)}{\partial y} \right) \hat{k}$$

$$= -2\hat{i} - 3x^2\,\hat{j} + 2y\,\hat{k}$$

Notice a couple of things here. First, the curl of a vector field is another vector field since it is defined by a cross product. And second, since we have variables left in our answer, the curl will generally vary from point to point. In any case, however, this particular vector field will tend to produce circulation. Below is graph of

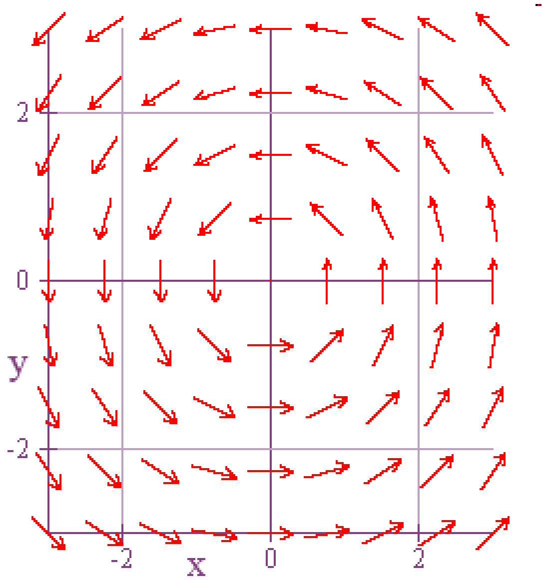
$\vec{F}(x,y,z) = -y^2\,\hat{i} + 2z\,\hat{j} + x^3\,\hat{k}$.



If we look at the two dimensional vector field $\vec{F}(x, y) = -y\hat{i} + x\hat{j}$, then the curl is,

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right) \hat{k} = 2\hat{k}.$$

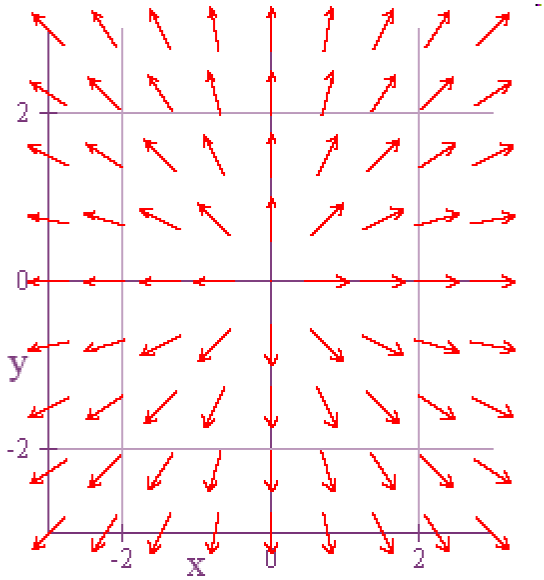
In this case, we see that the circulation is the same at every point in the field.



The scalar component of the curl of a two dimensional vector field is sometimes referred to as the scalar curl. Also, compare the result above with what happens with the curl of the vector field $\vec{F} = x\hat{i} + y\hat{j}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = \left(\frac{\partial y}{\partial x} - \frac{\partial (x)}{\partial y} \right) \hat{k} = 0\hat{k} = \vec{0}$$

The result here is that the scalar curl is zero, i.e. $\nabla \times \vec{F}$ is the zero vector. This corresponds to the fact that the vector field, as plotted below, has no tendency to produce circulation.



The other computation we want to learn now is the divergence of a vector field. This computation will tell us something about the tendency of a vector field to produce flux across a boundary. We define this quantity as follows.

If $\vec{F} = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$, then *divergence of* $\vec{F} = \nabla \bullet \vec{F}$

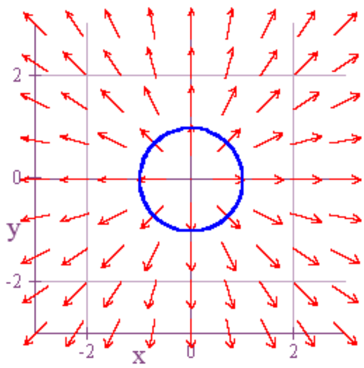
$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \bullet \left(P\hat{i} + Q\hat{j} + R\hat{k}\right) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

If $\vec{F}$ is a 2-dimensional vector field, $\vec{F} = P\hat{i} + Q\hat{j}$, then *divergence of* $\vec{F} = \nabla \bullet \vec{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$

Notice both that the divergence is easily defined in terms of a dot product and that the final result is a scalar or scalar function, not a vector. Let's use $\vec{F} = x\hat{i} + y\hat{j}$ as an example. In this case, the divergence is,

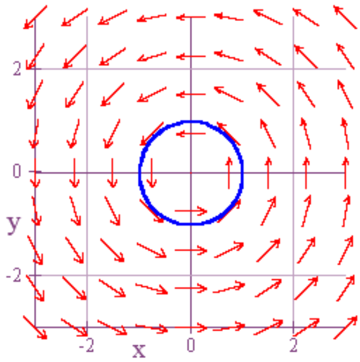
$$\nabla \bullet \vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 1 + 1 = 2$$

We can see that the result is a nonzero number, and that, in this particular example, the divergence is the same at all points in our plane. Let's now look at a graph of our vector field with a circle drawn about the origin.



It's very easy to see in this case that the vector field will tend to cause a flux across the boundary of this circle.

Now look at a similar plot for $\vec{F}(x, y) = -y\hat{i} + x\hat{j}$.



Does this vector field appear to be one that would cause flux across this boundary? Probably not, and so it should be no surprise that the divergence comes out equal to zero.

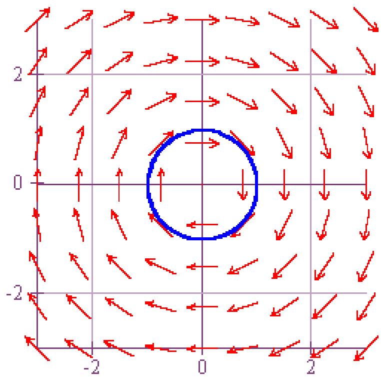
$$\nabla \bullet \vec{F} = \frac{\partial(-y)}{\partial x} + \frac{\partial x}{\partial y} = 0 + 0 = 0$$

A final word of caution, however. Even well drawn pictures can sometimes be a little misleading. Thus, the bottom line is always what the formulas tell us, not what you think the picture shows. A picture can be correct, but we can still misinterpret it.

Let's also look at two final examples of vector fields in two dimensions that are very instructive. We'll begin by computing the curl of the vector field $\vec{F} = y\hat{i} - x\hat{j}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \left(\frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y} \right)\hat{k} = -2\hat{k}$$

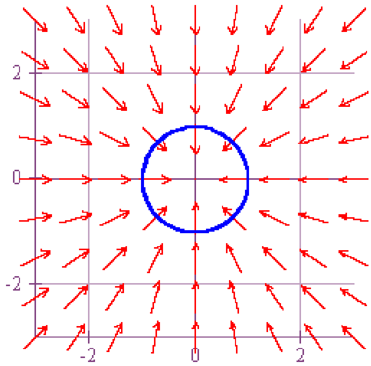
The scalar curl of this vector field is negative, and this corresponds to the fact that this field will tend to produce circulation in the clockwise rather than the counterclockwise direction. Because of this type of result, we always think of counterclockwise rotation as positive and clockwise as negative. Below is the graph of this vector field along with a circular curve with center at the origin. The vector field will circulate things in the clockwise direction around this curve.



And lastly, let's look at the divergence of the vector field $\vec{F} = -x\hat{i} - y\hat{j}$. The divergence is,

$$\nabla \bullet \vec{F} = \frac{\partial(-x)}{\partial x} + \frac{\partial(-y)}{\partial y} = -1 - 1 = -2$$

In this case, the divergence is negative, and we can see from the graph below how this result corresponds to a flux across our boundary that is towards the center of our circle rather than away from it.



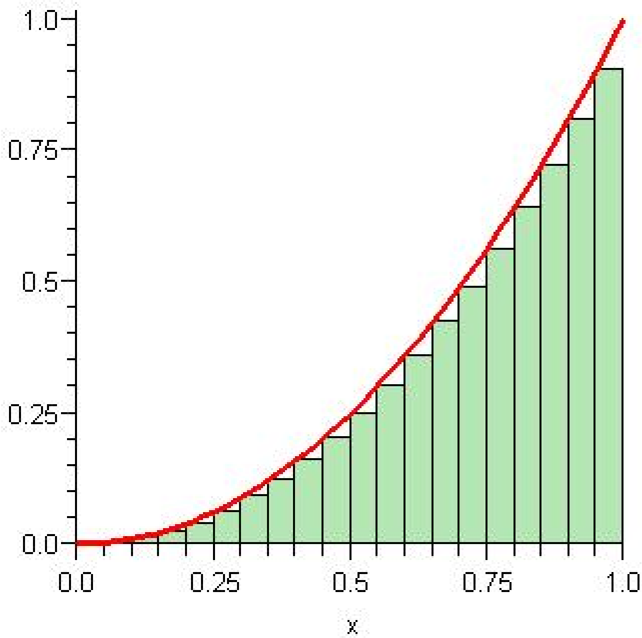
Next stop, line integrals!

CHAPTER 10

LINE INTEGRALS

I'm sure everyone has fond memories of integrating a function like $f(x) = x^2$ from $x = 0$ to $x = 1$ and getting $\frac{1}{3}$ as the exact area under the curve.

$$\int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$



What you didn't think about, probably, was that the interval you were integrating over was just a straight line, and that it might actually be possible to define integrals with respect to more arbitrary curves in space in a manner involving limits that's similar to what you did in previous calculus courses. And that's exactly what we're going to do! And we're going to call the result a "line integral."

<u>Definition:</u> If $f(x,y)$ is defined on a smooth curve C that is parametrized by $x = x(t)$ and $y = y(t)$, (or equivalently, $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$), where $a \le t \le b$, and if s represents arc length, then the line integral of $f(x,y)$ along C from the point $\left(x(a), y(a)\right)$ to the point $\left(x(b), y(b)\right)$ is,

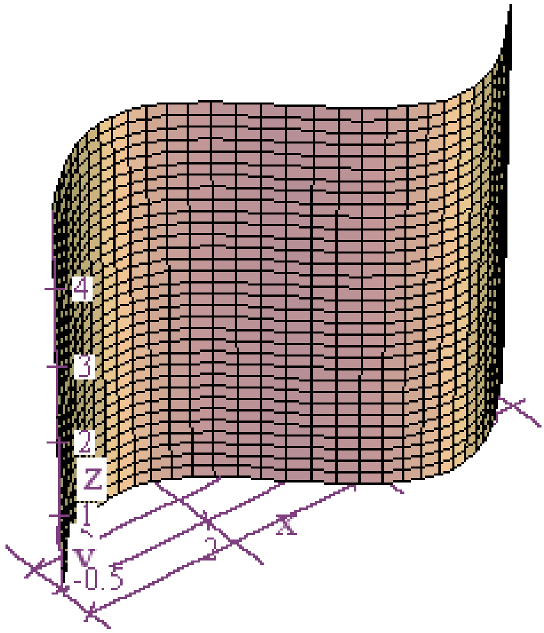
$$\int_C f(x,y)\,ds = \lim_{\Delta s \to 0} \sum f(x,y)\Delta s \text{, provided this limit exists.}$$

Suppose we have something like a curtain whose bottom is curled along a wavy path, and suppose $f(x,y)$ represents the height of the curtain at particular point. Then the line integral defined above would give us back the area of this curtain in the same way that integrals you've studied in the past have represented area.

Let's now talk about how we evaluate line integrals. In practice, we almost always use our parametrization for our curve to turn our line integral into an ordinary integral like the ones you dealt with in previous calculus courses. Using our parametrization, we get,

$$\int_C f(x,y)\,ds = \int_a^b f\left(x(t),y(t)\right)\cdot\frac{ds}{dt}dt = \int_a^b f\left(x(t),y(t)\right)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt\ .$$

Remember how a long time ago we talked about arc length $s = s(t)$ for a vector-valued function $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, and we saw that $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$? Well, here's where we are going to use that formula. To now make things a little more specific, let's suppose we have a curtain that is 5 feet high, but it's furled a bit so that it's base describes a sine curve going from 0 to 2π feet. What is the area of the curtain? In this case, just take $f(x,y) = 5$ and $\vec{r}(t) = t\,\hat{i} + \sin(t)\,\hat{j}$ where $0 \le t \le 2\pi$. Then our curtain will to look something like this,



And to find the area, just evaluate the line integral.

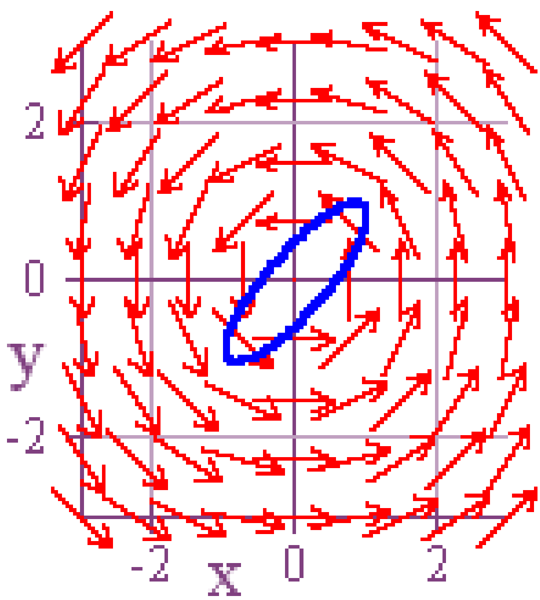
$$\int_C f(x,y)\,ds = \int_0^{2\pi} 5\cdot\frac{ds}{dt}\,dt = 5\int_0^{2\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = 5\int_0^{2\pi}\sqrt{1+\cos^2(t)}\,dt \approx 8.88\ ft^2$$

What's probably not obvious to you at this point is that we can integrate our function, if we wish, just with respect to *x* or solely with respect to *y* by using an appropriate parametrization. Here are the formulas for setting up those integrals.

$$\int_C f(x,y)\,dx = \int_a^b f\left(x(t),y(t)\right)\cdot\frac{dx}{dt}\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f\left(x(t),y(t)\right)\cdot\frac{dy}{dt}\,dt$$

These integrals with respect to *x* and *y*, far from being a mere mathematical abstraction, will actually be very important to us in the practical applications that follow. In particular, one of the most important applications of line integrals is to compute the work done by a vector field in pushing a particle along a curve. Thus, let's suppose we have a vector field $\vec{F}(x, y)$ and a smooth curve *C* that is parametrized in such a way that the curve is traced in the counterclockwise direction as *t* goes from $t = a$ to $t = b$. If the initial point of *C* coincides with the terminal point, then we'll also call *C* a closed curve, and if *C* doesn't intersect itself, except at the endpoints, then we'll call it simple. In general, we'll say that any simple, closed curve traced in a counterclockwise direction is positively oriented and any simple, closed curve traced in the clockwise direction is negatively oriented. Now recall that $\text{work} = \text{force} \times \text{distance}$.



Recall also that if our displacement is represented by a vector $\vec{D}$ and if the object displaced is acted upon by a force $\vec{F}$ pointing in a different direction, then the work done is equal to the component of $\vec{F}$ in the direction of the distance vector $\vec{D}$ times the length of $\vec{D}$. This gives us the following formula that we've seen before:

$$Work = comp_{\vec{D}}(\vec{F}) \bullet \|\vec{D}\| = \|\vec{F}\|\cos(\theta) \bullet \|\vec{D}\| = \|\vec{F}\|\|\vec{D}\|\cos(\theta) = \vec{F} \bullet \vec{D}$$

If our curve C is smooth and if the displacement of our particle is small, then as a result of local linearity, our displacement vector at a point is approximately equal to the change in arc length times the corresponding unit tangent vector. Hence,

$$Work \approx F \bullet (\Delta s \cdot T) = (F \bullet T) \cdot \Delta s$$

If we partition our curve C into a series of subintervals of length Δs, then the total work done by the force field in moving the particle along the curve C is:

$$Work \approx \sum (F \bullet T) \cdot \Delta s$$

$$\Rightarrow Work = \lim_{\Delta s \to 0} \sum (F \bullet T) \cdot \Delta s = \int_C F \bullet T \, ds$$

There are many different ways in which we can write this last formula. In particular,

$$\int_C F \bullet T\, ds = \int_C (F \bullet T)\frac{ds}{dt}dt = \int_C (F \bullet T)\|r'(t)\|dt = \int_C \left(F \bullet \frac{r'(t)}{\|r'(t)\|}\right)\|r'(t)\|dt$$

$$\int_C \left(F \bullet \frac{r'(t)}{\|r'(t)\|} \cdot \|r'(t)\|\right)dt = \int_C (F \bullet r'(t))\,dt = \int_C \left(F \bullet \frac{dr}{dt}\right)dt = \int_C F \bullet dr$$

This last expression, $\int_C \vec{F} \bullet d\vec{r}$, is how the line integral for work is often expressed.

Another formulation may be derived by writing $\vec{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$ and $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, where $a \le t \le b$. This gives us,

$$\int_C F \bullet T\, ds = \int_C F \bullet dr = \int_C \left(F \bullet \frac{dr}{dt}\right)dt$$

$$= \int_a^b \left(P\hat{i} + Q\hat{j}\right) \bullet \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}\right)dt = \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt}\right)dt = \int_C Pdx + Qdy$$

Now you can see why we pointed out that you can integrate line integrals just with respect to *x* or with respect to *y*. We will often want to write our work integral as $\int_C Pdx + Qdy$, and then use a parametrization for our curve to convert it into a more familiar integral that we know how to compute. For example, suppose our curve *C* is the unit circle parametrized in the counterclockwise direction by $\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j}$ with $0 \le t \le 2\pi$ and that our vector field is $\vec{F}(x,y) = -y\hat{i} + x\hat{j}$. Then the work done by this vector field in moving a particle around this curve is,

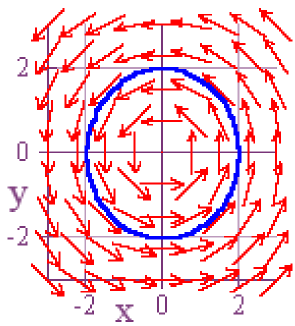
$$Work = \int_C \vec{F} \bullet d\vec{r} = \int_C P dx + Q dy = \int_a^b \left(-y \frac{dx}{dt} + x \frac{dy}{dt} \right) dt$$

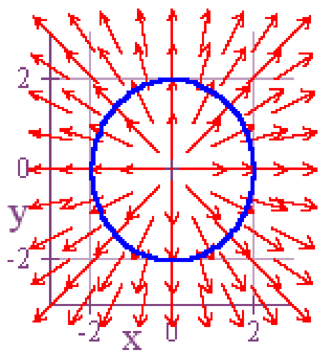
$$= \int_0^{2\pi} \left(-\sin(t)\left(-\sin(t)\right) + \cos(t)\cos(t) \right) dt = \int_0^{2\pi} \left(\sin^2 t + \cos^2 t \right) dt$$

$$\int_0^{2\pi} 1 dt = t \Big|_0^{2\pi} = 2\pi$$

Now let's go back for a minute to the formula $Work = \lim_{\Delta s \to 0} \sum (F \bullet T) \cdot \Delta s = \int_C F \bullet T \, ds$,

and let's contemplate it with respect to the images below.





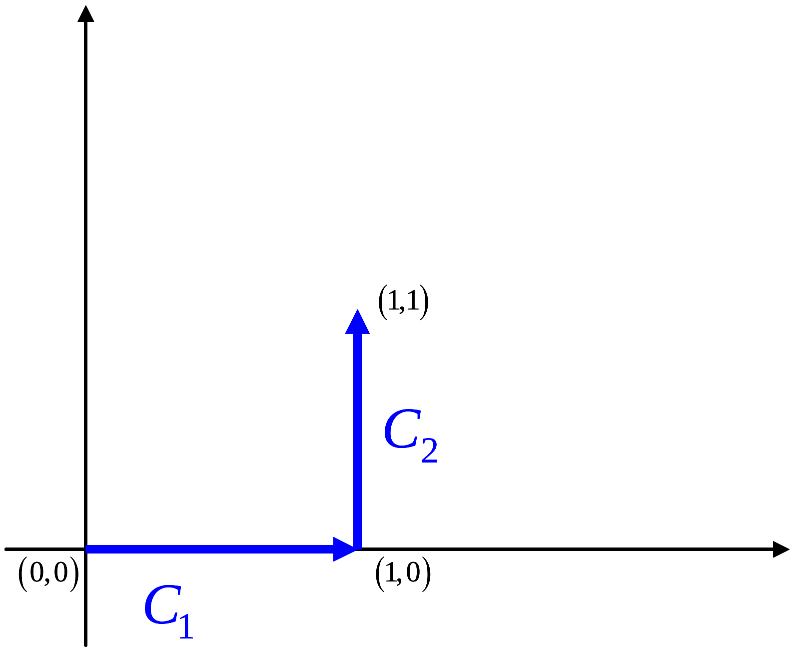
The dot product $\vec{F} \bullet T$ gives us the tangential component of the force when evaluated at a point on the circle above. In the first diagram, this tangential component is going to be quite a bit while in the second diagram, the tangential component is nil. Thus, for the first diagram the computation $\sum (F \bullet T) \cdot \Delta s$ will be much larger than what we would expect for the second. Also, when the tangential component of the force is substantial, then our vector field will tend to cause circulation about a curve as in the first diagram above, and when the tangential component of the force is completely lacking, then there will be no circulation created. The bottom line is that since $\lim_{\Delta s \to 0} \sum (F \bullet T) \cdot \Delta s = \int_C F \bullet T\, ds$, the same integral that measures the work done by a vector field along a given path also measures the circulation created by that vector field. In other words, $Circulation = \int_C F \bullet T\, ds = \int_C F \bullet dr = Work$.

Let's now look at a few more examples of line integrals. We'll include line integrals that involve circulation and work as well as more general line integrals. First, let's let our curve *C* be the circle of radius *r* with center at the origin parametrized by $\vec{r}(t) = r\cos(t)\hat{i} + r\sin(t)\hat{j}$ with $0 \le t \le 2\pi$. If we now evaluate $\int_C ds$, the integral along this curve with respect to arc length, then the result will simply be the length of our curve, and in particular, we will have found a simple way to derive the formula for the circumference of a circle.

$$\int_C ds = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt = \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{r^2\left(\sin^2 t + \cos^2 t\right)}\, dt = \int_0^{2\pi} \sqrt{r^2}\, dt = \int_0^{2\pi} r\, dt = rt\Big|_0^{2\pi} = 2\pi r$$

Now let's do a work integral where $\vec{F}(x, y) = x\hat{i} + y\hat{j}$, and our path is $C = C_1 + C_2$, the combination of the two straight line paths in the diagram below.



We can parametrize C_1 by $\vec{r}(t) = t\hat{i} + 0\hat{j} = t\hat{i}$ with $0 \le t \le 1$. Hence,

$$work_1 = \int_{C_1} \vec{F} \bullet d\vec{r} = \int_{C_1} Pdx + Qdy = \int_0^1 P\frac{dx}{dt}dt + Q\frac{dy}{dt}dt = \int_0^1 (t)(1) + (0)(0)\,dt = \left.\frac{t^2}{2}\right|_0^1 = \frac{1}{2}.$$

Similarly, we can parametrize C_2 by $\vec{r}(t) = 0\hat{i} + t\hat{j} = t\hat{j}$ with $0 \le t \le 1$. Thus,

$$work_2 = \int_{C_2} \vec{F} \bullet d\vec{r} = \int_{C_2} Pdx + Qdy = \int_0^1 P\frac{dx}{dt}dt + Q\frac{dy}{dt}dt = \int_0^1 (0)(0) + (t)(1)\,dt = \left.\frac{t^2}{2}\right|_0^1 = \frac{1}{2}$$

Therefore, the total work done is $work = \int_{C_1+C_2} \vec{F} \bullet d\vec{r} = \int_{C_1} \vec{F} \bullet d\vec{r} + \int_{C_2} \vec{F} \bullet d\vec{r} = \frac{1}{2} + \frac{1}{2} = 1.$

An important thing to realize in this example is that if we traversed our path C_2 in the opposite direction, then in our parametrization we could begin with $t=1$ and end with $t=0$, and this would result in the work along this path being equal to $-\frac{1}{2}$. Also, we can write this result as $\int_{-C_2} \vec{F} \bullet d\vec{r} = -\int_{C_2} \vec{F} \bullet d\vec{r}$. In other words, if you are doing a line integral and you reverse the direction in which you traverse your path, then all that this does is to change the sign of your original integral.

We now want to introduce a result for line integrals that is analogous to the Fundamental Theorem of Calculus, but before we do, we need to make some definitions. Remember when we said that you could always create a vector field by taking the gradient of a function of the sort $z = f(x, y)$? We'll it turns out that these kinds of vectors, i.e. gradient fields, are pretty special.

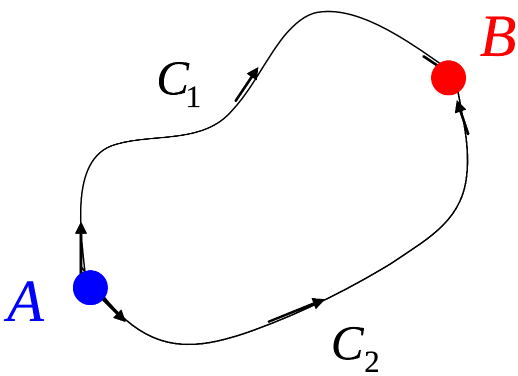
DEFINITION: A vector field $\vec{F} = P\hat{i} + Q\hat{j}$ is conservative if $\vec{F} = \nabla f$ for some function $z = f(x, y)$. In this case, f is called the potential function for $\vec{F}$.

DEFINITION: The line integral $\int_C \vec{F} \bullet d\vec{r}$ is independent of path if $\int_{C_1} \vec{F} \bullet d\vec{r} = \int_{C_2} \vec{F} \bullet d\vec{r}$ for any two paths C_1 and C_2 that have the same initial and terminal points.

Independence of path is going to mean that the value of the line integral depends only on the starting and stopping points of our curve. This should remind you of the Fundamental Theorem of Calculus where, when we have an antiderivative $F(x)$ of a continuous function $f(x)$, the value of the integral also depends only on the endpoints of our interval, $\int_a^b f(x)\,dx = F(b) - F(a)$. Our next step, though, is to give a condition that is equivalent to being independent of path.

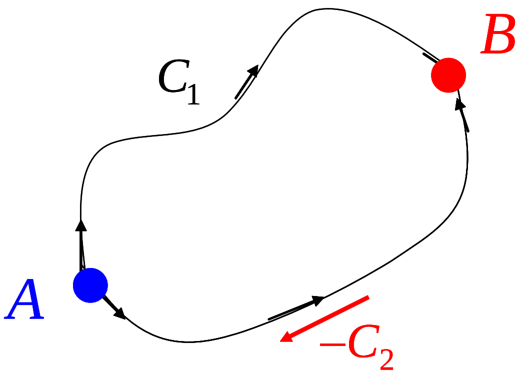
THEOREM: A line integral $\int_{C_1} \vec{F} \bullet d\vec{r}$ is independent of path in *D* if and only if $\int_C \vec{F} \bullet d\vec{r} = 0$ for every closed path C in *D*.

This actually pretty easy to see. Just consider the diagram below where we're trying to go from point *A* to point *B*.



If we have independence of path working for us, then $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$. However, if we reverse the direction along C_2, then $C = C_1 - C_2$ is a closed path, and $\int_C \vec{F} \cdot d\vec{r} = \int_{C_1 - C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = 0$. You can just as easily start with the assumption that an integral around any closed path C is equal to zero, decompose this path into $C = C_1 - C_2$, and conclude that $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$.

Case closed!



Now let's prove the Fundamental Theorem of Line Integrals.

The Fundamental Theorem of Lines Integrals: Let C be a smooth curve with a smooth parametrization $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ for $a \le t \le b$, and let $z = f(x, y)$ be a function whose gradient ∇f is continuous on C. Then,

$$\int_C \nabla f \cdot d\vec{r} = f\left(x(b), y(b)\right) - f\left(x(a), y(a)\right).$$

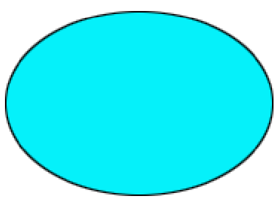
PROOF: $\int_C \nabla f \bullet d\vec{r} = \int_a^b \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} \right) \bullet \frac{d\vec{r}}{dt} dt = \int_a^b \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} \right) \bullet \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} \right) dt$

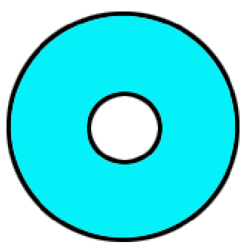
$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \right) = \int_a^b \frac{df}{dt} dt = f\left(x(b), y(b)\right) - f\left(x(a), y(a)\right).$

Well, that proof was short and sweet! What it is telling us, however, is that if our vector field is a gradient field, then the value of our line integral depends only upon the end points. We can do something analogous to the Fundamental Theorem of Calculus, i.e. find a type of antiderivative and evaluate. Notice, too, how, once again, the chain rule has been used to complete the proof. What we want do now, though, is figure out some conditions that will let us know that our line integral is independent of path. First, however, we need another definition.

DEFINITION: A region R is connected if any two points in R can be joined by a path C that lies in R. A connected region R is simply connected if it contains no holes.

Below are two connected regions, but only the first is simply connected.





And now, here are some things that are equivalent to being independent of path.

THEOREM: Let $\vec{F} = P\hat{i} + Q\hat{j}$ be a vector field defined on an open simply connected region R, let C be a smooth curve in R, and suppose P and Q have continuous first order derivatives in R. Then the following are equivalent.

1. $\vec{F} = P\hat{i} + Q\hat{j}$ is conservative.
2. $\vec{F} = \nabla f$ for some function $z = f(x, y)$.
3. $\int_C \vec{F} \bullet d\vec{r}$ is independent of path.
4. $\int_C \vec{F} \bullet d\vec{r} = 0$ for every closed curve C in R.
5. $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.
6. $curl\,\vec{F} = \nabla \times \vec{F} = \vec{0}$

The first four conditions are pretty obvious based upon what we've done so far. To see how number five works, suppose that $\vec{F} = P\hat{i} + Q\hat{j} = \nabla f$ for some function $z = f(x, y)$. Then $P = \dfrac{\partial f}{\partial x}$ and $Q = \dfrac{\partial f}{\partial y}$, and $\dfrac{\partial P}{\partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$ and $\dfrac{\partial Q}{\partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$. However, under the conditions above, these second order mixed partials are equal, and therefore, $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$. If we now rewrite this last equation as $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$, then this is just the statement that the scalar curl of $\vec{F}$ is zero. In other words, $curl\,\vec{F} = \nabla \times \vec{F} = \vec{0}$.

Finally, let's look at several examples to see how we can put this fundamental theorem into practice.

Example 1: $\vec{F} = 2x\hat{i} + 2y\hat{j}$ and C is any path from $(1,1)$ to $(2,2)$. Find $\int_C \vec{F} \bullet d\vec{r}$.

Solution: In this case, if $f(x,y) = x^2 + y^2$, then $\nabla f = \vec{F}$. Thus, the integral is independent of path and $\int_C \vec{F} \bullet d\vec{r} = f(2,2) - f(1,1) = 8 - 2 = 6$.

Example 2: If $\vec{F} = (3+2xy)\hat{i} + (x^2 - 3y^2)\hat{j}$, show that $\int_C \vec{F} \bullet d\vec{r}$ is independent of path.

Solution: Let $P = 3 + 2xy$ and $Q = x^2 - 3y^2$. Then $\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$. Therefore, $F = \nabla f$ for some function $z = f(x,y)$, and $\int_C \vec{F} \bullet d\vec{r}$ is independent of path.

Example 3: If $\vec{F} = (3+2xy)\hat{i} + (x^2 - 3y^2)\hat{j}$, find a potential function $z = f(x,y)$.

Solution: Let $P = 3 + 2xy$ and $Q = x^2 - 3y^2$. Then $\int P\,dx = 3x + x^2 y + g(y)$.

Differentiate this result with respect to y and you get $x^2 + g'(y)$. Comparing this result with $Q = x^2 - 3y^2$, we see that we want $g'(y) = -3y^2$. An antiderivative for this with respect to y is $-y^3$. Hence, it suffices to let $f(x,y) = 3x + x^2 y - y^3$.

Example 4: If $\vec{F} = (3+2xy)\hat{i} + (x^2 - 3y^2)\hat{j}$, find $\int_C \vec{F} \bullet d\vec{r}$ where the curve C is defined by $\vec{r}(t) = e^t \sin(t)\hat{i} + e^t \cos(t)\hat{j}$ where $0 \le t \le \pi$.

Solution: On this curve, $x = e^t \sin(t)$ and $y = e^t \cos(t)$. Also, the integral is independent of path, and a potential function for $\vec{F}$ is $f(x,y) = 3x + x^2 y - y^3$. Hence,

$$\int_C \vec{F} \bullet d\vec{r} = f\left(x(\pi), y(\pi)\right) - f\left(x(0), y(0)\right)$$

$$= f(e^\pi \sin\pi, e^\pi \cos\pi) - f(e^0 \sin(0), e^0 \cos(0))$$

$$= f(0, -e^\pi) - f(0,1)$$

$$= (3\cdot 0 + 0^2(-e^\pi) - (-e^\pi)^3) - (3\cdot 0 + 0^2(1) - 1^3)$$

$$= e^{3\pi} + 1.$$

Congratulations! If you've made it this far, then you've got potential!!!

CHAPTER 11

GREEN’S THEOREM, STOKES’ THEOREM, AND THE DIVERGENCE THEOREM

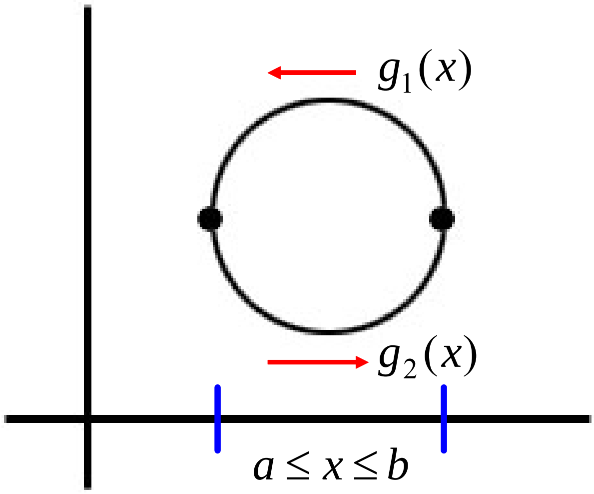
We’ve already seen one higher dimensional version of the Fundamental Theorem of Calculus, namely the Fundamental Theorem of Line Integrals. Now it’s time for another generalization of the FTC, Green’s Theorem. Hold on to your hats. This one’s a biggie!

Definition: A curve C is closed if its initial point coincides with its terminal point. A closed curve C is simple if it doesn’t cross or intersect itself except at its initial and terminal points.

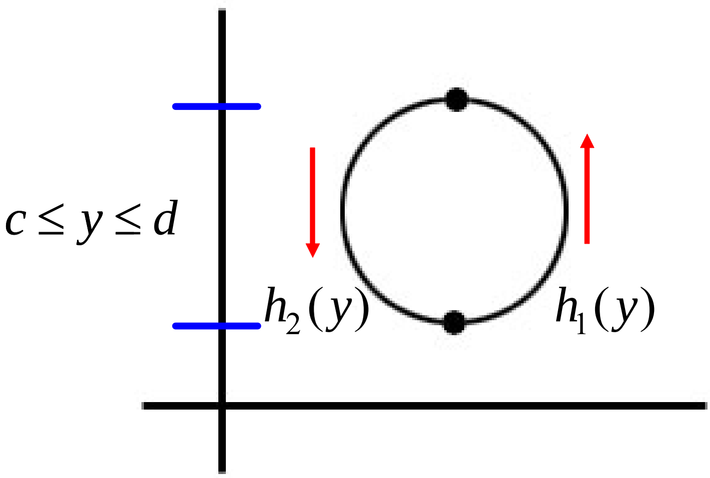
Green’s Theorem: Let C be a smooth, simple closed curve that is oriented counterclockwise in the xy-plane, let R be the region bounded by C and let $\vec{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$ be a vector field. If P and Q have continuous partial derivatives on an open region that contains R, then,

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

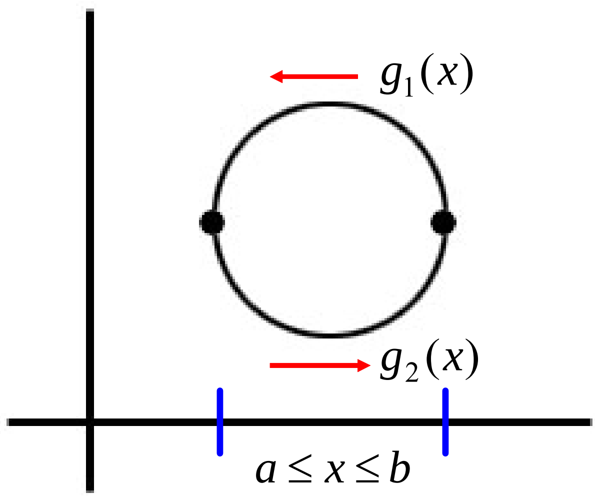
Proof: We'll do just a special case. Thus, suppose our counterclockwise oriented curve C and region R look something like the following:



In this case, we can break the curve into a top part and a bottom part over an interval on the x-axis from a to b, and we can denote the top part by the function $g_1(x)$ and the bottom part by $g_2(x)$. Or, we could just as easily portray x as varying from $h_2(y)$ to $h_1(y)$ as y varies from c to d.



Now let's begin. Suppose the curve below is oriented in the counterclockwise direction and is parametrized by *x*. Suppose also that the top part of our curve corresponds to the function $g_1(x)$ and the bottom part to $g_2(x)$ as indicated in the diagram below.



Then,

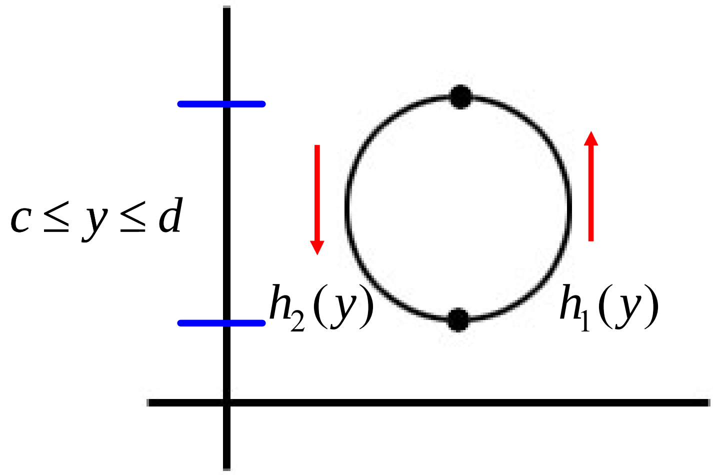
$$\int_C P\,dx = \int_a^b P(x, g_2(x))\,dx + \int_b^a P(x, g_1(x))\,dx$$
$$= \int_a^b P(x, g_2(x))\,dx - \int_a^b P(x, g_1(x))\,dx = \int_a^b P(x, g_2(x)) - P(x, g_1(x))\,dx$$

Notice that our integral of $P(x, g_1(x))$ goes from *b* to *a* since we are traversing the curve in the counterclockwise direction. Also, note the following,

$$-\iint_R \frac{\partial P}{\partial y}\,dA = -\int_a^b \int_{g_2(x)}^{g_1(x)} \frac{\partial P}{\partial y}\,dydx = -\int_a^b P(x, g_1(x)) - P(x, g_2(x))\,dx$$
$$= \int_a^b P(x, g_2(x)) - P(x, g_1(x))\,dx = \int_C P\,dx$$

Therefore, $\int_C P\,dx = -\iint_R \frac{\partial P}{\partial y}\,dA$.

In a similar manner, with respect to the diagram below, we can assert the following,



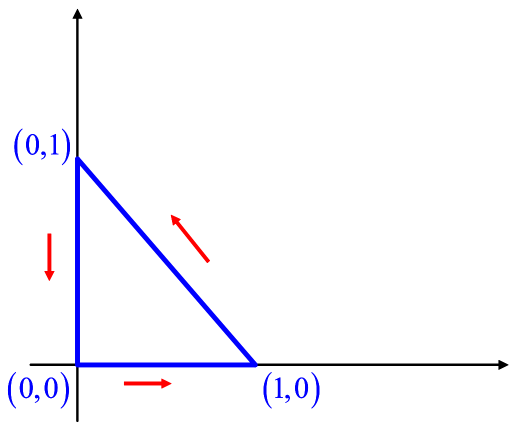
$$\iint_R \frac{\partial Q}{\partial x}\,dA = \int_c^d \int_{h_2(x)}^{h_1(x)} \frac{\partial Q}{\partial x}\,dxdy = \int_c^d Q(h_1(y), y) - Q(h_2(y), y)\,dy$$
$$= \int_c^d Q(h_1(y), y)\,dy + \int_d^c Q(h_2(y), y)\,dy = \int_C Q\,dy$$

Therefore, $\int_C Q\,dy = \iint_R \frac{\partial Q}{\partial x}\,dA$, and thus, $\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$. ■

Here is the really remarkable thing that Green's Theorem is saying. We have a region *R* that is bounded by a curve *C*, and Green's Theorem is telling us the value of the double integral of *R* depends entirely upon what happens on the boundary curve *C*. This is exactly what is going on in the Fundamental Theorem of Calculus and the Fundamental Theorem of Line Integrals. The only difference is that in the earlier theorems we dealt with a line instead of a region, and the boundary of the line is represented by its endpoints instead of a curve. Except for the change in dimension, the results are analogous. This is also something that we continue to see as we move higher up into theoretical mathematics. Quite often it's what happens on the boundary that determines things. However, in one respect, this should not surprise us. After all, aren't most things in life defined by their boundary? On physical plane, for example, the boundary of my body defines where I end and the rest of the world begins. Boundaries create the distinctions that result in individual existence. Thus, in a larger sense, these theorems are not so surprising.

Now it's time for an example!

Example 1: Evaluate $\int_C x^4\,dx + xy\,dy$ where C is the positively oriented triangle defined by the line segments connecting $(0,0)$ to $(1,0)$, $(1,0)$ to $(0,1)$, and $(0,1)$ to $(0,0)$.



Solution: By changing the line integral along C into a double integral over R, the problem is immensely simplified.

$$\int_C x^4\,dx + xy\,dy = \int_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_0^{1-x} (y-0)\,dy dx$$

$$= \int_0^1 \frac{y^2}{2}\bigg|_0^{1-x} dx = \int_0^1 \frac{(1-x)^2}{2} dx = \frac{-(1-x)^3}{2\cdot 3}\bigg|_0^1 = \frac{1}{6}$$

Now let's look at another problem which can be greatly simplified by applying Green's Theorem.

Example 2: Evaluate $\int_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$ where C is the circle $x^2 + y^2 = 9$.

Solution: Again, Green's Theorem makes this problem much easier.

$$\int_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy = \int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_R \left[\frac{\partial(7x + \sqrt{y^4+1})}{\partial x} - \frac{\partial(3y - e^{\sin x})}{\partial y} \right] dA$$

$$= \iint_R (7-3)\,dA = \int_0^{2\pi}\int_0^3 4r\,dr\,d\theta = \int_0^{2\pi} 2r^2\Big|_0^3\,d\theta$$

$$= \int_0^{2\pi} 18\,d\theta = 18\theta\Big|_0^{2\pi} = 36\pi$$

Now let's look at a couple of generalizations of Green's Theorem, namely Stokes' Theorem and the Divergence Theorem. In two dimensions, it's very easy to see that these are both simply different ways of looking at Green's Theorem, and so we'll start with the two dimensional versions of these results. Also, we'll finally see in these theorems why the definitions that we gave for curl and divergence wind up telling us something about the circulation and flux of a vector field along or across a curved path.

Let's begin by supposing that we have a vector field $\vec{F} = P\hat{i} + Q\hat{j}$ and that $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ is a smooth parametrization for a curve *C*. Then Green's Theorem and previous results tells us that,

$$\text{Work } = \int_C \vec{F} \bullet d\vec{r} = \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

However, recall that this same integral is also a measure of the circulation around he curve caused by the vector field. Additionally, notice that the expression $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$ is how we defined the scalar component of the curl for a two dimensional vector field. Thus,

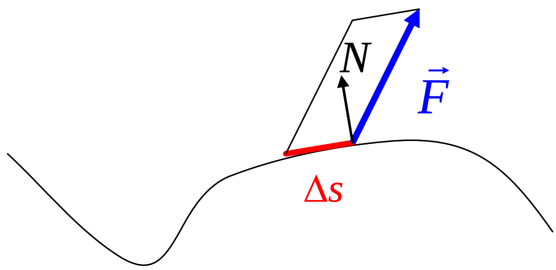
$$\begin{aligned} \text{Circulation } &= \int_C \vec{F} \bullet T\,ds = \int_C \vec{F} \bullet d\vec{r} = \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \iint_R \left(curl\,\vec{F} \right) \bullet \hat{k}\,dA = \iint_R \left(\nabla \times \vec{F} \right) \bullet \hat{k}\,dA \end{aligned}$$

This last result that $\text{Circulation } = \int_C \vec{F} \bullet d\vec{r} = \iint_R \left(curl\,\vec{F} \right) \bullet \hat{k}\,dA$ is known as the two dimensional version of Stoke's Theorem.

To develop the two dimensional version of the Divergence Theorem, recall that if

$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ with $a \le t \le b$ is parametrization for *C*, then the unit tangent is defined as $T(t) = \frac{\vec{r}'(t)}{\|r'(t)\|} = \frac{x'(t)}{\|r'(t)\|}\hat{i} + \frac{y'(t)}{\|r'(t)\|}\hat{j}$, and the unit normal is $N(t) = \frac{y'(t)}{\|r'(t)\|}\hat{i} - \frac{x'(t)}{\|r'(t)\|}\hat{j}$.

Now consider the diagram below.



What we want to think about is how a force $\vec{F}$ might push material across this curve. In this case, think of $\vec{F}$ as a velocity vector. Additionally, if our change in arc length, Δs, is small, then we can treat our velocity vector as if it were constant over this interval. In this case, the amount of material or flux across this boundary in a unit of time is going to be approximately equivalent to the area of the parallelogram defined by the Δs and the vector $\vec{F}$. To get the height of this parallelogram, we just take the dot product of $\vec{F}$ with the unit normal vector N, and then we multiply this result by the base Δs. In other words, $\textit{flux across } \Delta s = \textit{area} = \left(\vec{F} \bullet N\right)\Delta s$. To now get the total flux across the curve C, we just sum up all the individual fluxes and take a limit as Δs goes to zero. In other words,

$$Flux = \lim_{\Delta s \to 0} \sum \left(\vec{F} \bullet N \right) \Delta s = \int_C \left(\vec{F} \bullet N \right) ds$$

Now, as you might suspect, there are some other ways in which we can write this integral, and it's going to be a direct result of applying Green's Theorem.

$$\begin{aligned}
\text{Flux } &= \int_C \vec{F} \bullet N \, ds = \int_a^b \left(\vec{F} \bullet N \right) \frac{ds}{dt} dt \\
&= \int_a^b \left(P\hat{i} + Q\hat{j} \right) \bullet \left(\frac{y'(t)}{\|\vec{r}'(t)\|} \hat{i} - \frac{x'(t)}{\|\vec{r}'(t)\|} \right) \|\vec{r}'(t)\| \, dt \\
&= \int_a^b \left(P\hat{i} + Q\hat{j} \right) \bullet \left(\frac{dy}{dt} \hat{i} - \frac{dx}{dt} \right) dt = \int_a^b \left(P\frac{dy}{dt} - Q\frac{dx}{dt} \right) dt \\
&= \int_C Pdy - Qdx = \int_C -Qdx + Pdy = \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA \\
&= \iint_R div \, \vec{F} \, dA = \iint_R \left(\nabla \bullet \vec{F} \right) dA
\end{aligned}$$

If you look at this proof closely, you'll see that it uses a lot of tools that we've developed throughout this book, and you'll also see at the very end why our definition of the divergence does indeed tell us something about flux across a boundary.

The Divergence Theorem is also known as Gauss' Theorem, and below we have a summary of our results for the two dimensional case.

If $\vec{F} = P\hat{i} + Q\hat{j}$ is a vector field and if C is a simple closed, counterclockwise oriented path parametrized by $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, and if T is the unit tangent vector and N is the unit normal vector, then:

GREEN'S THEOREM: Work $= \int_C \vec{F} \bullet d\vec{r} = \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

STOKES' THEOREM: Circulation $= \int_C \vec{F} \bullet T\,ds = \int_C \vec{F} \bullet d\vec{r}$

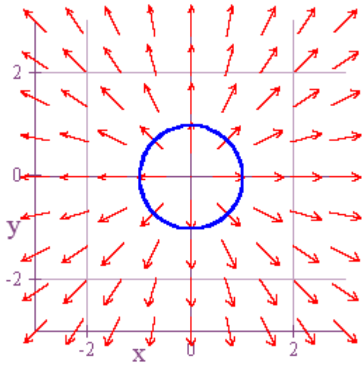
$$= \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R \left(curl\,\vec{F} \right) \bullet \hat{k}\,dA = \iint_R \left(\nabla \times \vec{F} \right) \bullet \hat{k}\,dA$$

GAUSS' THEOREM: Flux $= \int_C \vec{F} \bullet N\,ds == \int_C Pdy - Qdx = \int_C -Qdx + Pdy$

$$= \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA = \iint_R div\,\vec{F}\,dA = \iint_R \left(\nabla \bullet \vec{F} \right) dA$$

Let's now look at an example or two before moving on.

<u>Example 3:</u> Find the circulation and flux of $\vec{F} = x\hat{i} + y\hat{j}$ with regard to the unit circle below. Assume a positive (counterclockwise) orientation for the curve.

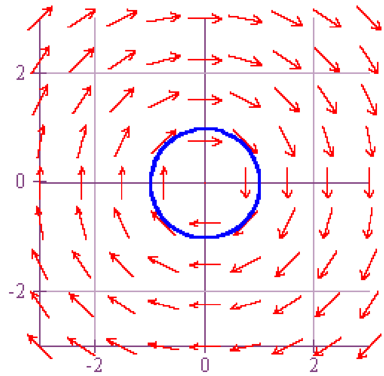


<u>Solution:</u> Our initial suspicion should be that the circulation is zero and that the flux is positive since our vectors are pointing away from the center of the circle. Calculations will confirm this.

$$\text{Circulation } = \int_C \vec{F} \bullet d\vec{r} = \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) dA$$
$$= \iint_R (0-0)\,dA = 0$$

$$\text{Flux } = \int_C \vec{F} \bullet N\,ds = \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA = \iint_R \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} \right) dA = \iint_R (1+1)\,dA$$
$$= 2\iint_R dA = 2 \cdot (\text{area of the circle}) = 2\pi$$

Example 4: Find the circulation and flux of $\vec{F} = y\hat{i} - x\hat{j}$ with regard to the unit circle below. Assume a positive (counterclockwise) orientation for the curve.



Solution: This time our suspicion should be that the circulation is negative since the vectors suggest a rotation in the clockwise or negative direction, and that the flux is zero.

$$\text{Circulation } = \int_C \vec{F} \bullet d\vec{r} = \int_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R \left(\frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y} \right) dA$$

$$= \iint_R (-1-1)\, dA = -2 \iint_R dA = -2 \cdot (\textit{area of the circle}) = -2\pi$$

$$\text{Flux } = \int_C \vec{F} \bullet N\, ds = \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA = \iint_R \left(\frac{\partial y}{\partial x} + \frac{\partial(-x)}{\partial y} \right) dA$$

$$= \iint_R (0+0)\, dA = 0$$

Now let's look at Stokes' Theorem in three dimensions. We'll basically do just the case that is easiest to understand. In particular, we'll assume that we have a smooth surface $z = f(x,y)$ that is bounded by a curve *C*, and we'll assume that the domain of $z = f(x,y)$ is a nice, simply connected region *R* in the *xy*-plane bounded by a curve C_R that, of course, gets mapped onto the curve *C* by our function $z = f(x,y)$. We'll also assume that $\vec{F}(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$ is a vector field that has continuous partial derivatives on an open region in $\mathbb{R}^3$ (three dimensional coordinate space) that contains the surface *S*. As usual, continuity prevents anything really unusual, bad, or unexpected from happening, and given these conditions, we can succinctly state our theorem as,

<u>Stokes' Theorem:</u> $\int_C \vec{F} \bullet d\vec{r} = \iint_S \left(curl\,\vec{F} \bullet N \right) dS$.

<u>Proof:</u> Let's talk first about the vector *N*. What the heck is that? We'll as you might suspect, *N* is a unit normal vector, but in this case we're talking about a vector being normal or perpendicular to the surface $z = f(x,y)$. So how do we find our unit normal? Again, we already know one way to do it. Remember that we can write $z = f(x,y)$ as $0 = f(x,y) - z$, and we can consider the surface *S* to be a level surface of the function $g(x,y,z) = f(x,y) - z$. Consequently, as you surely recall, the gradient of

g is $\nabla g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} - \hat{k}$, and ∇g is normal to the surface at any surface point we wish to evaluate it at. Now realize this. If ∇g is perpendicular to the surface, then $-\nabla g$ is also perpendicular to the surface. Right? It just points in the opposite direction as ∇g. Also, $-\nabla g$ is the vector we are going to use because since $-\nabla g = -\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}$ has a positive $\hat{k}$ component, it will point upward with respect to our surface, and that is going to be much nicer. Now, how do we get a unit normal vector out of his? Simple! We just divide $-\nabla g$ by its length. Thus,

$$N = \frac{-\nabla g}{\left\| -\nabla g \right\|} = \frac{-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}.$$

Now recall that when we did a surface integral a long time ago as an example of a double integral, we saw that $dS = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dA$. Consequently,

$$\iint_S \left(curl\,\vec{F} \bullet N\right) dS = \iint_R \left[curl\,\vec{F} \bullet \frac{-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}} \right] \cdot \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dA$$

$$= \iint_R \left[curl\,\vec{F} \bullet \left(-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k} \right) \right] dA$$

Now we're making progress! The next step is to remember that

$$curl\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}.$$ Thus,

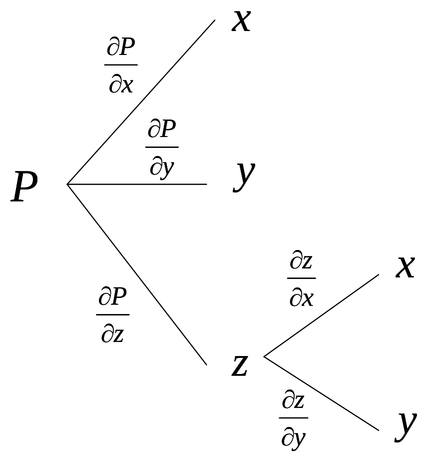
$$\iint_S \left(Curl\,F \bullet N\right) dS = \iint_R \left[curl\,\vec{F} \bullet \left(-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k} \right) \right] dA$$
$$= \iint_R \left[\left(\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k} \right) \bullet \left(-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k} \right) \right] dA$$
$$= \iint_R \left(-\frac{\partial R}{\partial y}\frac{\partial f}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial P}{\partial z}\frac{\partial f}{\partial y} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$
$$= \iint_R \left(-\frac{\partial z}{\partial x}\frac{\partial R}{\partial y} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Also, $\int_C \vec{F} \bullet d\vec{r} = \int_C Pdx + Qdy + Rdz$. So, all we have to do is to show that

$$\int_C Pdx + Qdy + Rdz = \iint_R \left(-\frac{\partial z}{\partial x}\frac{\partial R}{\partial y} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$ Is that too

much to ask? Certainly not! Let's begin.

There are two things we need now. First, let's suppose that our curve C_R is nicely parametrized by $x = x(t)$ and $y = y(t)$ for $a \le t \le b$. Then this also provides us with a parametrization for *C* if we let $z = z(t) = f\left(x(t), y(t)\right)$. Second, we're going to have to use some unusual versions of the chain rule, and so let's look at the instructive diagrams below. For the function *P*, our diagram is,



Hence, if we want the derivative of *P* with respect to *y*, then we multiply along all the branches that terminate in *y* and add 'em up. This gives us,

$$\text{derivative of } P \text{ with respect to } y = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y}.$$

If you're really sharp, you might notice that there is something a little funny about what I just wrote. In particular, I wrote out the phrase "derivative of *P* with respect to *y*" instead of using the symbolic notation $\frac{\partial P}{\partial y}$. However, if I had used this notation, then I would have gotten the equation

$$\frac{\partial P}{\partial y} = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y}$$

which would imply that

$$0 = \frac{\partial P}{\partial z}\frac{\partial z}{\partial y},$$

and that is not at all what we want! So what's going on here? Well, basically, our notation is failing us a bit, so let's make up an example and follow it through. Suppose we have,

$$\begin{aligned} &P = y^3 + z^2 \quad \& \\ &z = y^4 \end{aligned}$$

Then what we are really trying to say when we write

$$\text{derivative of } P \text{ with respect to } y = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y}$$

is,

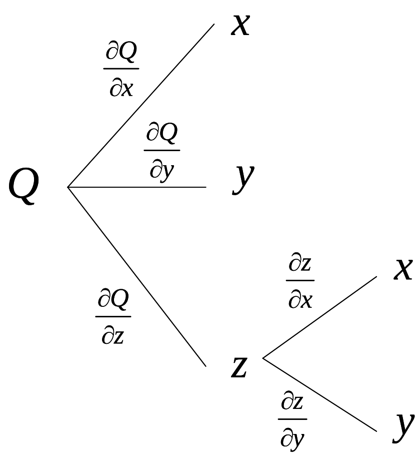
$$\begin{aligned} &\text{derivative of } P \text{ with respect to } y \\ &\quad = \text{derivative of the part that is explicitly written in terms of } y \\ &\qquad + \text{derivative of the part that is explicity written in terms of } z \\ &\qquad\quad \times \text{derivative of } z \text{ with respect to } y \end{aligned}$$

Thus, to find $\frac{\partial P}{\partial y}$ in this example, we first find the derivative of y^3 with respect to y, and then we apply the chain rule to z^2. The end result is,

$$\frac{\partial P}{\partial y} = \frac{\partial(y^3)}{\partial y} + \frac{\partial(z^2)}{\partial z}\frac{\partial z}{\partial y} = 3y^2 + 2z \cdot \frac{\partial(y^4)}{\partial y} = 3y^2 + 2y^4 \cdot 4y^3 = 3y^2 + 8y^7$$

Make sense? Good! Now let's continue with the proof.

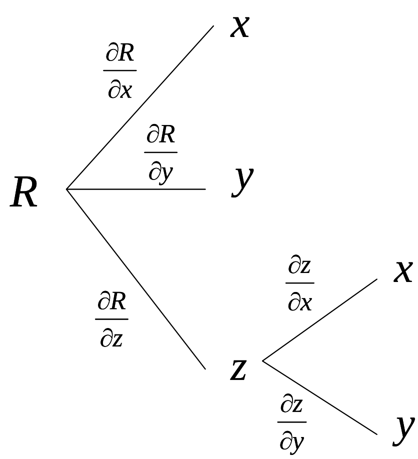
For our function *Q*, the diagram is,



And,

$$\text{derivative of } Q \text{ with respect to } x = \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x}.$$

Finally, the diagram for *R* is,



Furthermore,

$$\text{derivative of } R \text{ with respect to } x = \frac{\partial R}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}$$

And,

$$\text{derivative of } R \text{ with respect to } y = \frac{\partial R}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}$$

Now we’re ready to rock-n-roll! Just make the above substitutions into the calculations below when the time comes.

$$\int_C \vec{F} \bullet d\vec{r} = \int_C Pdx + Qdy + Rdz = \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right) dt$$

$$= \int_a^b \left[P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \right) \right] dt = \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{\partial z}{\partial x}\frac{dx}{dt} + R\frac{\partial z}{\partial y}\frac{dy}{dt} \right) dt$$

$$= \int_a^b \left[\left(P + R\frac{\partial z}{\partial x} \right)\frac{dx}{dt} + \left(Q + R\frac{\partial z}{\partial y} \right)\frac{dy}{dt} \right] dt = \int_{C_R} \left(P + R\frac{\partial z}{\partial x} \right) dx + \left(Q + R\frac{\partial z}{\partial y} \right) dy$$

$$= \iint_R \left[\frac{\partial\left(Q + R\frac{\partial z}{\partial y} \right)}{\partial x} - \frac{\partial\left(P + R\frac{\partial z}{\partial x} \right)}{\partial y} \right] dA$$

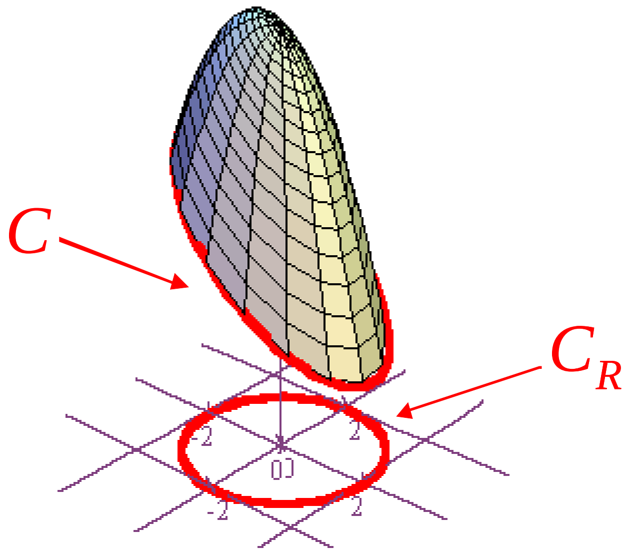
$$= \iint_R \left[\begin{array}{l} \left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} \right) + \left(R\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y}\left[\frac{\partial R}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x} \right] \right) \\ \qquad - \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} \right) - \left(R\frac{\partial^2 z}{\partial y \partial x} + \frac{\partial z}{\partial x}\left[\frac{\partial R}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y} \right] \right) \end{array} \right] dA$$

$$= \iint_R \left(\begin{array}{l} \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y}\frac{\partial R}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial R}{\partial z}\frac{\partial z}{\partial x} - \frac{\partial P}{\partial y} - \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} \\ \qquad - R\frac{\partial^2 z}{\partial y \partial x} - \frac{\partial z}{\partial x}\frac{\partial R}{\partial y} - \frac{\partial z}{\partial x}\frac{\partial R}{\partial z}\frac{\partial z}{\partial y} \end{array} \right) dA$$

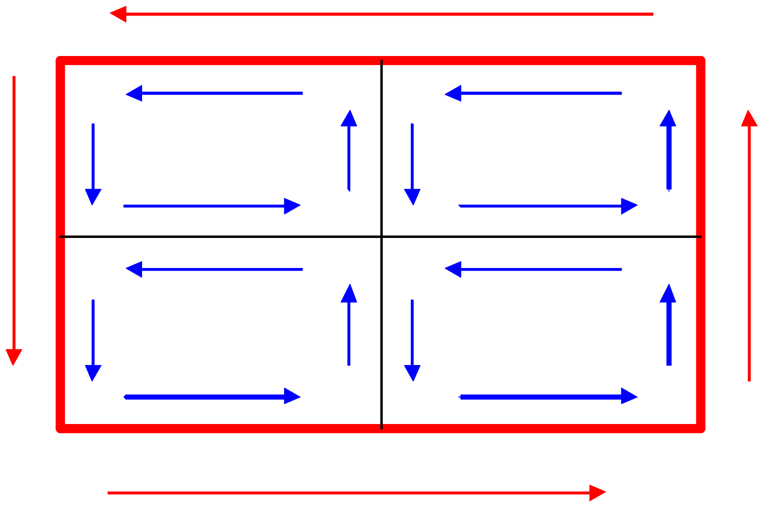
$$= \iint_R \left(-\frac{\partial z}{\partial x}\frac{\partial R}{\partial y} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_S \left(curl\, \vec{F} \bullet N \right) dS \qquad \blacksquare$$

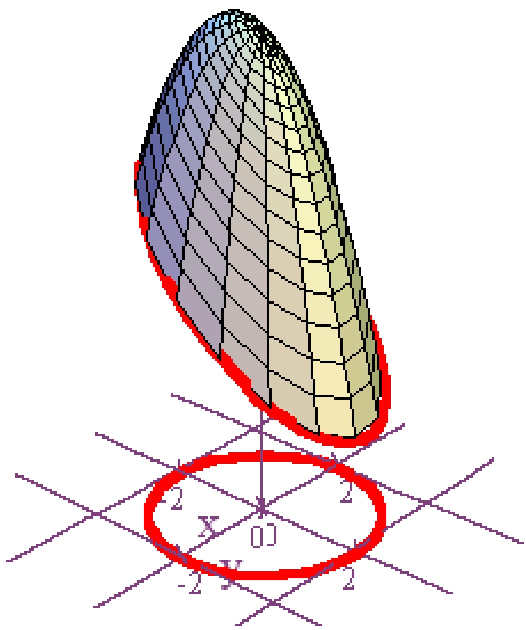
Wow! What a great proof! I'm really stoked! Now let's look at a few pictures in order to get another perspective on what Stokes' Theorem is saying.



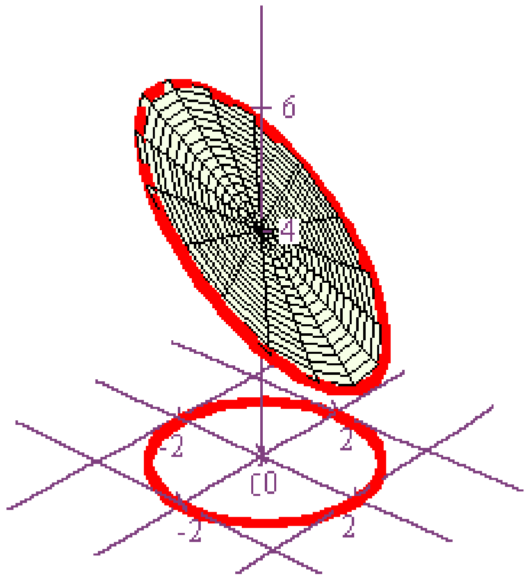
Above is a picture of a surface S that is bounded by a curve C. Right below C there is a corresponding curve C_R in the xy-plane that bounds a region R. Then the proof of Stokes' Theorem essentially shows us that the circulation integral around C is equivalent to a double integral of the curl over S which is equivalent to a double integral of the region R directly below S which is equivalent to a circulation integral around the curve C_R in the xy-plane. The bottom line of all these conclusions, though, is simply that the double integral of our curl over S is equivalent to the circulation integral around C. This is also somewhat easy to intuit from the following diagram.



Suppose we are trying to do a line integral over all the oriented paths represented by the blue arrows. Then the net result is going to be that some of these integrals are going to cancel each other out because for one integral our blue arrow will be pointing in one direction and for another integral our arrow will be pointing in the opposite direction. In fact the only paths that don't cancel are those represented by the red arrows. A similar thing happens with surfaces in three dimensions. In the graph below, if you imagine doing a line integral in the counterclockwise direction around each little polygon that we are using to help depict our surface, then once again lots of things will cancel out, and we'll be left only with a line integral around the bounding curve *C* at the bottom of the surface. What this is basically showing us is that an integral over the surface is approximately equivalent to a sum of integrals over the polygons used to depict that surface which is equivalent, by an extended version of Green's Theorem, to a sum of line integrals around the boundaries of those polygons which in turn reduces to a single line integral around the curve that bounds our surface. Case closed!



Notice, too, that the result of Stokes' Theorem does not depend so much on what our surface looks like. It only depends on the bounding curve. Thus, for a given vector field, the diagram below will give the same result as the one above.



Now let's see if we can do an example. We'll let our surface be the top half of a sphere of radius 1. More precisely, let $z = \sqrt{1 - x^2 - y^2}$ and let our vector field be $\vec{F} = -y\hat{i} + x\hat{j} + z\hat{k}$. Then the bounding curve C is the unit circle in the *xy*-plane, and we can think of our surface as a level surface for the function

$g = -f(x, y) + z = -(1 - x^2 - y^2)^{1/2} + z$. Furthermore,

$$\begin{aligned}\nabla g &= -\frac{1}{2}(1 - x^2 - y^2)^{-1/2}(-2x)\hat{i} - \frac{1}{2}(1 - x^2 - y^2)^{-1/2}(-2y)\hat{j} + \hat{k} \\ &= \frac{x}{\sqrt{1 - x^2 - y^2}}\hat{i} + \frac{y}{\sqrt{1 - x^2 - y^2}}\hat{j} + \hat{k}\end{aligned}.$$

Also,

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} = 0\hat{i} - 0\hat{j} + (1+1)\hat{k} = 2\hat{k}\,.$$

Thus, by Stokes' Theorem,

$$\int_C \vec{F} \bullet d\vec{r} = \iint_S \left(curl\, \vec{F} \bullet N \right) dS = \iint_R \left(\nabla \times \vec{F} \right) \bullet \nabla g\, dA$$

$$= \iint_R 2\hat{k} \bullet \left(\frac{x}{\sqrt{1-x^2-y^2}} \hat{i} + \frac{y}{\sqrt{1-x^2-y^2}} \hat{j} + \hat{k} \right) dA = \iint_R 2\, dA = 2\pi$$

If we try and do this by evaluating the line integral on the left side more directly, then we begin with $\int_C \vec{F} \bullet d\vec{r} = \int_C Pdx + Qdy + Rdz$ where C is the unit circle that may be parametrized by,

$$\begin{aligned} x &= \cos t \\ y &= \sin t \\ z &= 0 \\ 0 &\le t \le 2\pi \end{aligned}$$

And,

$$\begin{aligned} P &= y = -\sin t \\ Q &= x = \cos t \\ R &= 1 \end{aligned}$$

Additionally,

$$\frac{dx}{dt} = -\sin t$$
$$\frac{dy}{dt} = \cos t$$

Thus,

$$\int_C \vec{F} \bullet d\vec{r} = \int_C P dx + Q dy + R dz = \int_0^{2\pi} P\frac{dx}{dt}dt + Q\frac{dy}{dt}dt + R\frac{dz}{dt}dt$$
$$= \int_0^{2\pi} (-\sin t)(-\sin t) + (\cos t)(\cos t) + 0 \; dt$$
$$= \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt = \int_0^{2\pi} dt = t\Big|_0^{2\pi} = 2\pi$$

So there you go. We get the same answer either way.

Now let's take a look at the Divergence Theorem.

<u>Divergence Theorem:</u> Let *V* be a solid region bounded by a closed surface *S* and let *N* be a unit normal vector pointing outward from the solid *V*. If $\vec{F} = P\hat{i} + Q\hat{j} + Rk$ is a vector field who component functions have continuous partial derivatives throughout *V*, then $\iint_S \vec{F} \bullet N \, dS = \iiint_V \left(\nabla \bullet \vec{F}\right) dV$.

Proof: We want to show that

$\iint_S \vec{F}\bullet N\,dS = \iint_S \left(P\hat{i}\bullet N + Q\hat{j}\bullet N + Rk\bullet N\right)dS = \iiint_V \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dV$. Furthermore, it will suffice to show that,

$$\iint_S \left(P\hat{i}\bullet N\right)dS = \iiint_V \frac{\partial P}{\partial x}dV,$$

$$\iint_S \left(Q\hat{j}\bullet N\right)dS = \iiint_V \frac{\partial Q}{\partial y}dV, \text{ and}$$

$$\iint_S \left(R\hat{k}\bullet N\right)dS = \iiint_V \frac{\partial R}{\partial z}dV$$

We'll prove just the last equality, $\iint_S \left(R\hat{i}\bullet N\right)dS = \iiint_V \frac{\partial R}{\partial z}dV$, since the other proofs are similar. Of course, what a professor generally means when he or she says this is,

1. I've never even tried to prove the other cases.
2. I don't even know if I can do the other cases.
3. Oh, please, oh, please, don't make me try and do the other cases!!!

But, continuing on, let's suppose our solid has a top surface S_2 defined by $z = f_2(x,y)$, and a bottom surface S_1 defined by $z = f_1(x,y)$, and maybe some vertical sides. If we set $g_2 = -f(x,y)+z$ and $g_1 = f(x,y)-z$, then ∇g_2 and ∇g_1 will both be outward pointing vectors that are perpendicular to our surface *S*. We don't need to really worry about the vertical sides because for any unit normal N at such a side point we automatically have $R\hat{k}\bullet N = 0$. With this setup we now have,

$$\iint_S \left(R\hat{k}\bullet N\right)dS = \iint_{S_2}\left(R\hat{k}\bullet N\right)dS + \iint_{S_1}\left(R\hat{k}\bullet N\right)dS$$
$$= \iint_R R(x,y,f_2(x,y))\hat{k}\bullet\left[-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}\right]dA + \iint_R R(x,y,f_1(x,y))\hat{k}\bullet\left[\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} - \hat{k}\right]dA$$
$$= \iint_R R(x,y,f_2(x,y)) - R(x,y,f_1(x,y))\,dA$$
$$= \iint_R\left[\int_{f_1(x,y)}^{f_2(x,y)}\frac{\partial R}{\partial z}dz\right]dA = \iiint_V \frac{\partial R}{\partial z}dV. \qquad \blacksquare$$

And there you go! Slam dunk!! The proof is similar for the other parts (so they say), and when you add everything together you get $\iint_S \vec{F}\bullet N\,dS = \iiint_V\left(\nabla\bullet\vec{F}\right)dV$.

And now it's time to make an example of this theorem! We'll use the same volume and vector field we used last time, but this time we'll use the Divergence Theorem to find the flux across the boundary. So once again, let our surface be the top half of a sphere of radius 1, $z = \sqrt{1-x^2-y^2}$, and let our vector field be $\vec{F} = -y\hat{i} + x\hat{j} + z\hat{k}$. Then the corresponding region R in the xy-plane is the unit circle. And as before, we can think of our surface as a level surface for the function

$g = -f(x,y) + z = -(1-x^2-y^2)^{1/2} + z$. Now, using the Divergence Theorem, we have,

$$\iint_S \vec{F} \bullet N \, dS = \iiint_V \left(\nabla \bullet \vec{F} \right) dV = \iiint_V dV = \frac{2\pi}{3}$$

If we now try to integrate our surface integral a little more directly, then we get,

$$\iint_S \vec{F} \bullet N \, dS = \iint_R \left(\vec{F} \bullet \nabla g \right) dA = \iint_R \left(\frac{-xy}{\sqrt{1-x^2-y^2}} + \frac{xy}{\sqrt{1-x^2-y^2}} + z \right) dA$$

$$= \iint_R z \, dA = \iint_R \sqrt{1-(x^2+y^2)} \, dA = \int_0^{2\pi} \int_0^1 \sqrt{1-r^2} \, r dr d\theta = \left[\text{set } u = 1-r^2, \, du = -2rdr \right]$$

$$= -\frac{1}{2} \int_0^{2\pi} \int_1^0 u^{1/2} \, du d\theta = -\frac{1}{2} \int_0^{2\pi} \left[\frac{2u^{3/2}}{3} \Bigg|_1^0 \right] d\theta = \int_0^{2\pi} \frac{1}{3} d\theta = \frac{\theta}{3} \Bigg|_0^{2\pi} = \frac{2\pi}{3}$$

We got the same answer twice, so it must be right. Adios, for now. It's been a pleasure!

Christopher Benton is from parts unknown and grew up Deepinthehearta, Texas. After many adventures and misadventures, he finally became an adult and obtained a master of science degree and a doctor of philosophy degree in mathematics from the University of Houston. He currently lives in Arizona in The Valley of the Sun with his beloved wife, Susan, and their dog, Dr. Chloe Continuum, Do.G. (doctor of growlology). Chloe's favorite pastimes are eating tasty dog food and curling up every evening with a good calculus book to read.