DOC BENTON'S FORBIDDEN SECRETS OF MULTIVARIABLE CALCULUS

VOLUME I

Functions of Several Variables, Cylindrical and Spherical Coordinates, Parametric Equations for Curves in Space, Vectors, Vector Transformations, and Vector Calculus

CHRISTOPHER P. BENTON, PHD

Dedicated To

שושן דודי

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INTRODUCTION

Why is this multivariable calculus book different from all other multivariable calculus books? Well, for one thing, it's free! Frankly, I'm tired of all those textbooks that cost a small fortune to purchase. Education should be affordable for everyone!

Another thing that makes this book different is that it is not really meant to be a textbook or to replace whatever book you are using in class. Instead, it's meant to be a story, the story of multivariable calculus, and like all good stories it has a beginning and an end and the various parts of the story are all interconnected with one another. That is one of the things I wanted to show here, how multivariable calculus is a grand adventure where tools developed early on become critical parts of the story that follows. Additionally, I've left out some topics (with regret) and expanded others (with glee!) simply to make the story flow a little better. I don't cover everything that one might cover in multivariable calculus, but I do cover a lot and I do enough to give you a good basis for what it's all about.

And lastly, this book is different because I've created a lot of online resources that are available for anyone to use for free! These include problem sets that also double as additional examples, graphers and calculators for functions of several variables that allow constructions and explorations that go far beyond anything that was available when I first took calculus in the early seventies, and PowerPoint slide presentations that go along with this book and sometimes even cover things that I haven't included in this book. To access this material, just go to www.docbenton.com, and look for links to my multivariable calculus courses and stuff. And above all, enjoy!

-Doc Benton

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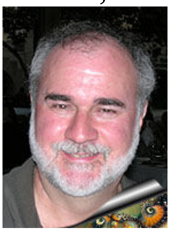


CHAPTER 1

FUNCTIONS OF SEVERAL VARIABLES

Congratulations! If you're reading this book, then you've probably passed both Calculus I and Calculus II, and you're ready to step into a calculus universe of multiple dimensions. Enjoy the trip! This book is about calculus of several variables, and that means that we will deal with situations that could, if we so wish, involve three, four, or even hundreds or thousands of variables and dimensions. However, since visualizing things in four or more dimensions is rather daunting, we'll generally restrict ourselves to the usually three.

Our journey begins with a discussion and exploration of functions of several variables, so let's stop for a moment and think about just what we mean by that. Since we are talking about functions, that means that a unique input should generate a single output. Additionally, since we are talking about functions of several variables, that suggests that the input will consist of two or more variables rather than the usual one variable such as x. A function of one variable, for example, might be written as

$$y = 2x + 3$$

or,

$$f(x) = x^2.$$

A single input, in each of the cases above, is followed by a single output. In contrast, a function of two variables might appear as

$$z = f(x, y) = x^2 - 2xy + 5$$

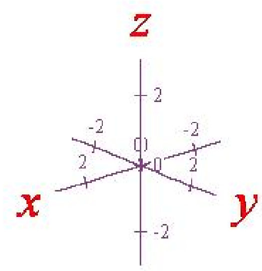
or,

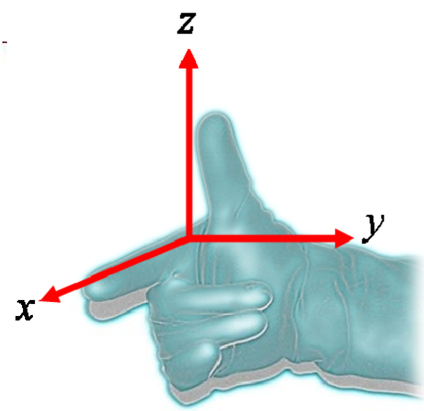
$$z = 2x + 3y - 1.$$

In each of these functions, two variables are used to determine the output.

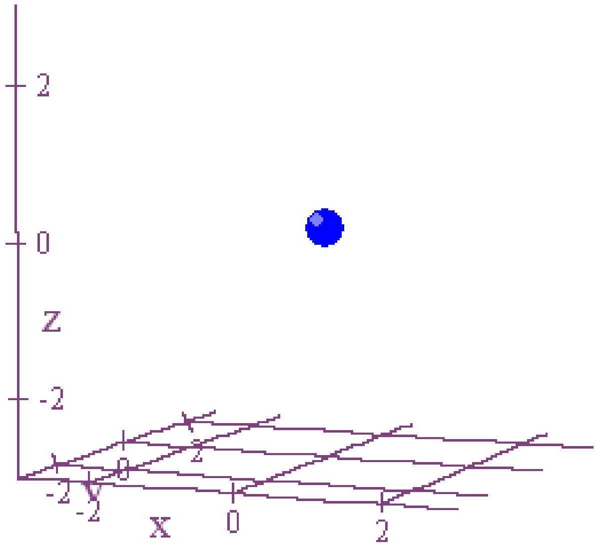
Our first goal is going to be to understand the graphs of functions of two variables, but before we engage that topic, ask yourself if you were ever taught functions of several variables in the past. If your memory is good, then you'll understand that the answer is yes, and, in fact, you were probably first introduced to functions of this sort in elementary school! For example, when you were told that $Area = Length \times Width$, or that $Distance = Rate \times Time$, you were being given a function of two variables. Think about it! Each of the above formulas takes two input values, and any two particular input values result in a single output value. That makes it a function, and since we have two inputs, it's a function of two variables! Thus, you've really been studying functions of two variables for a long time. You just didn't know it.

To graph a function of the form $y = f(x)$ we need two dimensions, one for the input and one for the output. Thus, it should come as no surprise that to plot a function of the form $z = f(x, y)$ we will need three dimensions, two for the inputs and one for the output. Generally, to plot points in three dimensions we use what we call a right-hand coordinate system. This means that if you take your right hand and point your index finger in the direction of the positive *x*-axis and your middle finger in the direction of the positive *y*-axis, then your thumb will point in the direction of the positive *z*-axis. These days virtually everyone uses a right-hand coordinate system, and a given point is located in space by coordinates (x, y, z)





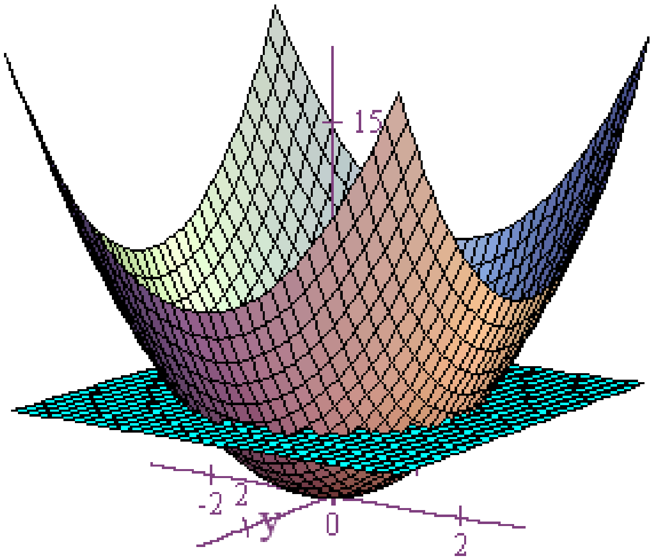
Additionally, the mathematical software that is available today often gives us several options regarding how our axes are displayed. In fact, a very common mode of display is to frame our graph by placing the axes on the sides of the image. Below is an example where all three axes are off to the side, and each variable ranges from -3 to 3. A right-hand coordinate system is being used, but the positive *x*-axis points towards the right of the image while the positive *y*-axis is directed towards the back. A single point $(0,0,0)$ has also been plotted to help you understand where the origin is in this type of presentation.

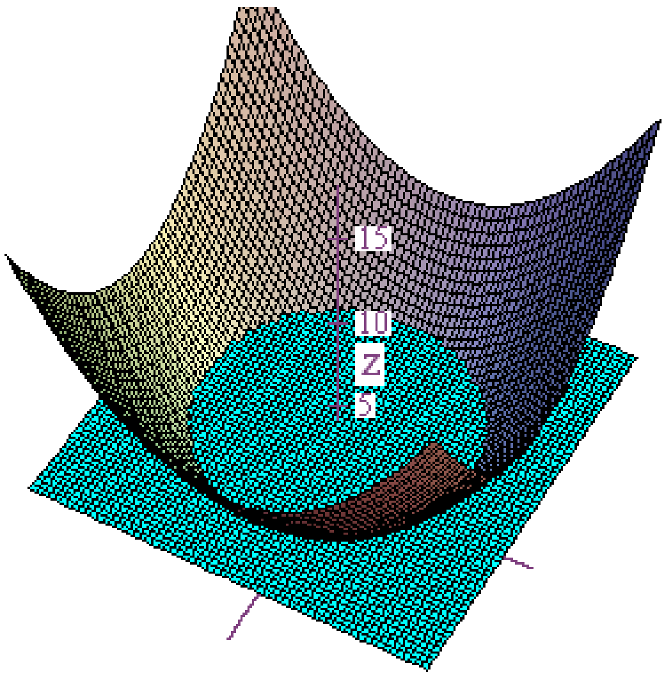


Now let's take a typical function of two variables and think about how we can analyze it. In particular, let's start with $z = f(x, y) = x^2 + y^2$. What can we say about this function? Well, it's not hard to see that $z = f(x, y) = x^2 + y^2$ always has to be greater than or equal to zero. In fact, the only time we will have $z = 0$ is when $x = 0$ and $y = 0$. This point, denoted by $(x, y, z) = (0,0,0)$, is also the lowest point on the graph. Furthermore, as we move away from the origin in any direction, $z = f(x, y) = x^2 + y^2$ has to get larger. This already gives us the image of some surface that has its lowest point at $(0,0,0)$ and whose sides go up higher and higher as we move away from that low point.

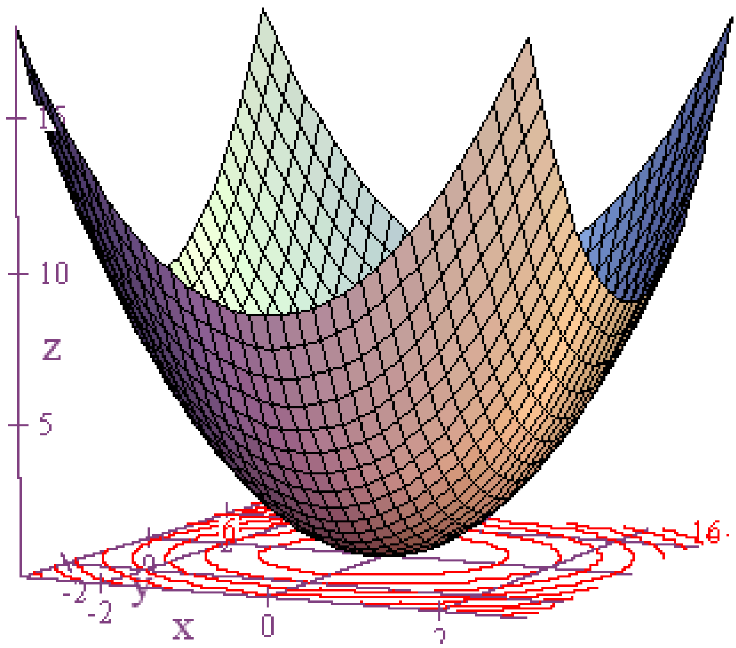
Now let's think about some more sophisticated ways in which we might analyze the graph of $z = f(x, y) = x^2 + y^2$. This next method is what I call "slicing and dicing." For instance, think of taking a cross-section of this surface by fixing a value of *z* such as $z = 4$. If we fix *z* to 4, then our equation becomes $x^2 + y^2 = 4$, and this is an equation for a circle of radius 2 with center at the origin. Similarly, if we fix *z* to any value whatsoever that is larger than zero, we will likewise get a circular cross-section. Now try and visualize an object that has its bottom point at $(0,0,0)$, rises upward as you move away from this point, and has circular cross-sections when you slice through it by fixing the value of *z*.

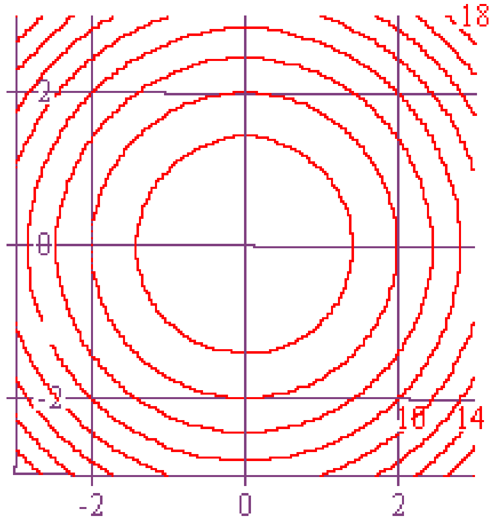
When we get a cross-section of our surface by fixing a value of *z*, we generally call that cross-section a contour or level curve. Some people like to call it a contour if the curve is graphed on the surface itself at the appropriate elevation, and then they call it a level curve if the cross-section has been moved down into the *xy*-plane. I'll always follow this convention except for when I don't. To give an example, below is a graph of what happens when we intersect the graph of $z = f(x, y) = x^2 + y^2$ with the plane $z = 4$. We can easily see that the cross-section is a circle by tilting the graph and looking down from the top.



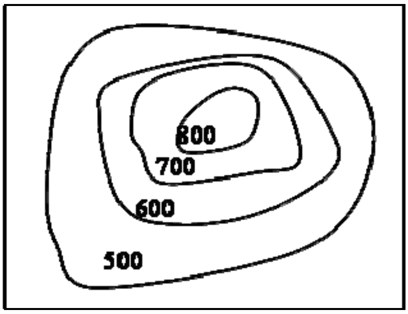


On the other hand, if we move several of our contours down into the *xy*-plane, then the resulting level curve diagram looks like this.


10
z
5
16
x
0


18
2
0
-2
10
14
-2
0
2

This latter diagram is basically the same thing as a topographic map. A topographic map such as the one below for a small hill gives us a 2-dimensional representation that helps us more easily visualize the corresponding 3-dimensional surface. In a similar manner, level curve diagrams also allow us to more easily capture in our minds the corresponding 3-dimensional image.



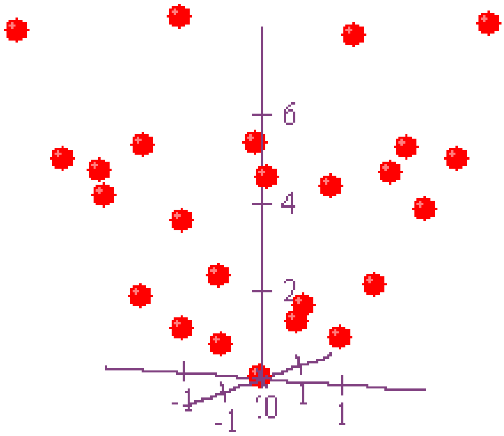
Small Hill

We can do more slicing and dicing by fixing *x* and *y* to particular values. For example, if we fix $y = 2$, then our cross-section is $z = x^2 + 4$ which is an equation for a parabola that opens upward. Likewise, if we fix $x = 2$, then we get $z = 4 + y^2$ which is another equation for an upward opening parabola. Thus, while cross-sections of $z = f(x, y) = x^2 + y^2$ created by fixing *z* are circular, cross-sections created by fixing *x* or *y* will be parabolas. For this reason, the graph of $z = f(x, y) = x^2 + y^2$ is usually called a *paraboloid*.

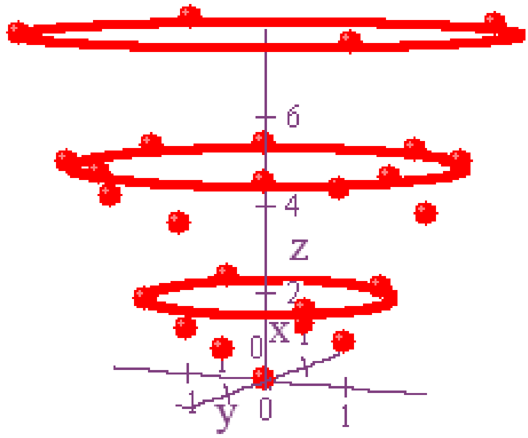
Up to this point, we've focused on how to analyze and visualize just parts of a graph, but soon we will shift our focus to the entire graph itself. First, though, let's do what we normally do when we graph a function of one variable such as $y = f(x)$. Let's create a table of values in order to give us some points to plot! Below is a table of values for $z = f(x, y) = x^2 + y^2$ where the *x*-values go from top to bottom on the left side while the *y*-values are spaced left to right along the top. Both sets of values in this table range from -5 to 5. Additionally, in each individual cell you will find the *z*-value corresponding to the given *x* and *y*. Then end result is as follows.

x\y	**-5**	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**	**5**
-5	50	41	34	29	26	25	26	29	34	41	50
-4	41	32	25	20	17	16	14	20	25	32	41
-3	34	25	18	13	10	9	10	13	18	25	34
-2	29	20	13	8	5	4	5	8	13	20	29
-1	26	17	10	5	2	1	2	5	10	17	26
0	25	16	9	4	1	0	1	4	9	16	25
1	26	14	10	5	2	1	2	5	10	14	26
2	29	20	13	8	5	4	5	8	13	20	29
3	34	25	18	13	10	9	10	13	18	25	34
4	41	32	25	20	17	16	14	20	25	32	41
5	50	41	34	29	26	25	26	29	34	41	50

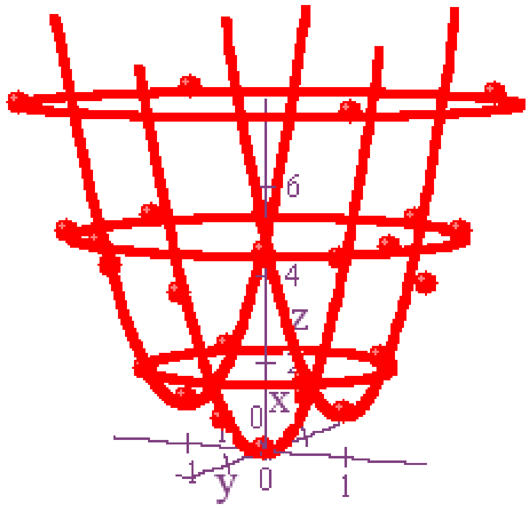
If we now plot the points in the table we created, here's what we get.



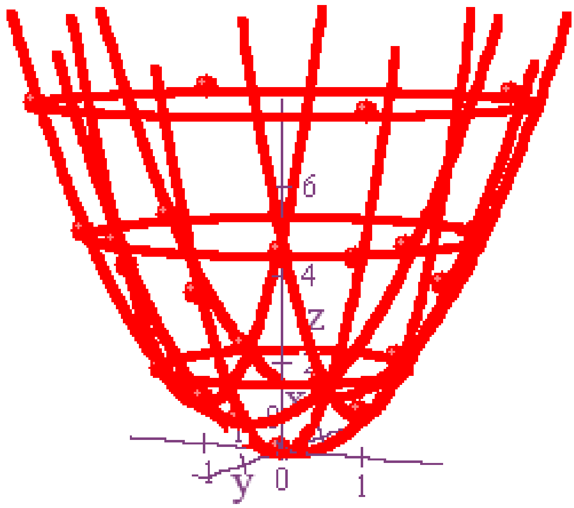
The good news is, we've got a plot! The bad news is, it still doesn't look like much. We just lose too much perspective when we try to plot three dimensional points in two dimensions. However, this is where slicing and dicing can help us. For example, let's look at what happens when we generate level curves by setting $z = 2, 5,$ or 8. The results are equations for circles of radii $\sqrt{2}, \sqrt{5},$ and $\sqrt{8}$, (i.e., $x^2 + y^2 = 2,\ x^2 + y^2 = 5,$ and $x^2 + y^2 = 8$). If we add these circles at the appropriate elevation to our graph, then we get the following.



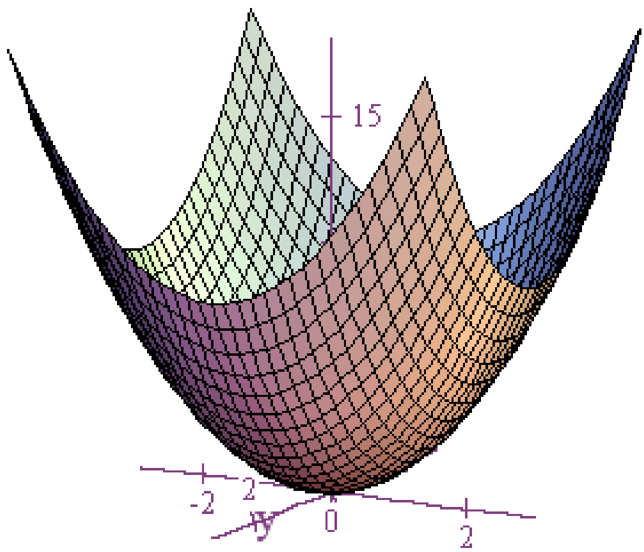
This certainly makes the graph a little easier to understand. However, let's add to the perspective even more by looking at the cross-sections that we get by setting *x* and *y* equal to fixed values. First, we'll set $y = -1, 0,$ and 1. This gives us the equations $z = x^2 + 1$, $z = x^2$, and $z = x^2 + 1$. Add the graphs of these curves in the positions corresponding to $y = -1, 0,$ and 1, and we get the following.



Now we're making some progress! Let's refine our efforts even more by graphing the cross-sections that we get by setting $x = -1, 0,$ and $1,$ corresponding to $z = 1 + y^2$, $z = y^2$, and again $z = 1 + y^2$. Here's what we get.



Now it's really beginning to take shape! However, to tell the truth, we never really do things quite this way. After all, if we have software that can do the kinds of graphs of cross-sections that we've done above, then we can certainly use that software to simply draw the complete graph to begin with, and when we do, we get something like this.



The point, though, is that we don't always have such software available, and even if we do, it still deepens our understanding if we are able to do some analysis of the function before creating a computer generated image of the surface. For example, regarding the graph of $z = f(x, y) = x^2 + y^2$, we were able to figure out the following.

1. The z-values are always greater than or equal to zero.
2. The lowest point is at $(0,0,0)$.
3. The values of z increase as we move further away from the origin.
4. The cross-sections obtained by setting z equal to a constant are circles.
5. The cross-sections obtained by setting x or y equal to constant values are parabolas.

By doing this kind of analysis, we can get a pretty good idea beforehand of what the surface graph will look like. And by the way, since many of the cross-sections of

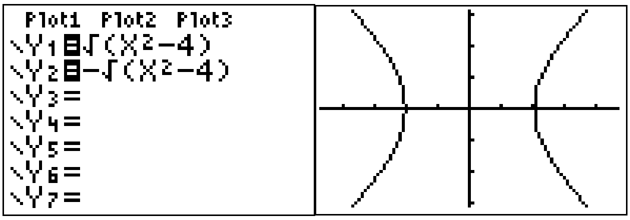
$z = f(x, y) = x^2 + y^2$ are parabolas, this is, again, why this surface is referred to as a *paraboloid*. It's basically our 3-dimensional version of a parabola, and it's often one of the first surfaces that multivariable calculus students encounter.

Now let's look at some other examples. Each one, in addition to being a standard example of a function of two variables, illustrates an important point in the analysis of such functions.

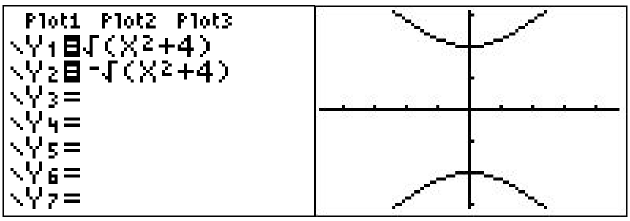
Example 1: $z = f(x, y) = x^2 - y^2$

Like the previous function we looked at, the output value is zero when both of the inputs are zero. However, unlike the graph of $z = x^2 + y^2$, this function will take on both positive and negative values. We can see this by setting, in turn, $x = 0$ and $y = 0$. If $x = 0$, then the equation reduces to $z = -y^2$ which is a parabola opening downward. Thus, as we now let *y* vary, the values for *z* can take on any possible negative value. However, if instead we set $y = 0$, then we get the equation $z = x^2$, and now *z* can take on any possible positive value as we let *x* vary. Furthermore, notice that if you fix *y* to any value *k*, then the cross-sections described as $z = x^2 - k^2$ are all parabolas opening upward. Similarly, if we fix $x = k$, then the cross-sections given by $z = k^2 - y^2$ are parabolas opening downward.

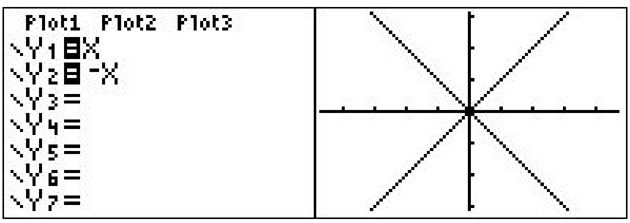
Now, what about the level curves obtained by setting $z = k$? Let's explore a few using a TI-84 graphing calculator. For instance, if we set $z = 4$, then we get the equation $4 = x^2 - y^2$. If we try to solve this for *y*, then we first get $y^2 = x^2 - 4$. From here we have to take into account both positive and negative square roots to obtain $y = \pm\sqrt{x^2 - 4}$. In order to graph this on our calculator, we have to split it up into two functions, $y_1 = \sqrt{x^2 - 4}$ and $y_2 = -\sqrt{x^2 - 4}$.



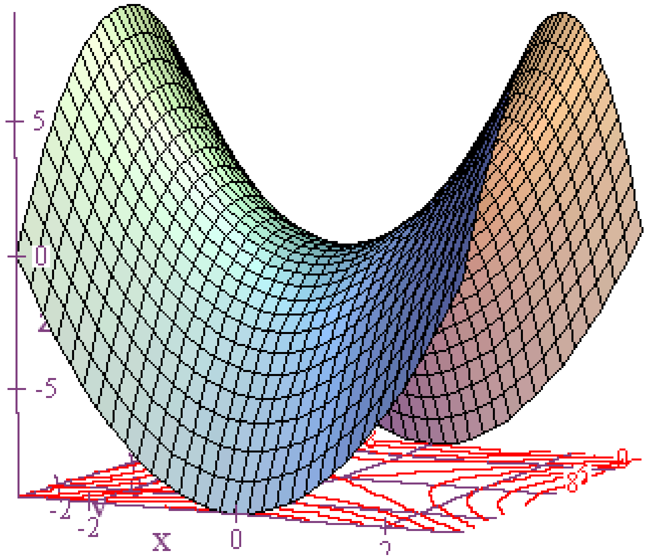
The shape of the graph is a hyperbola, and some of you may have recognized right off the bat that $4 = x^2 - y^2$ is an equation for a hyperbola that crosses the *x*-axis at -2 and 2. Similarly, if we set $z = -4$, then we get the equation $-4 = x^2 - y^2$ which is another hyperbola. However, this time we have *y*-intercepts of -2 and 2, and no *x*-intercepts.

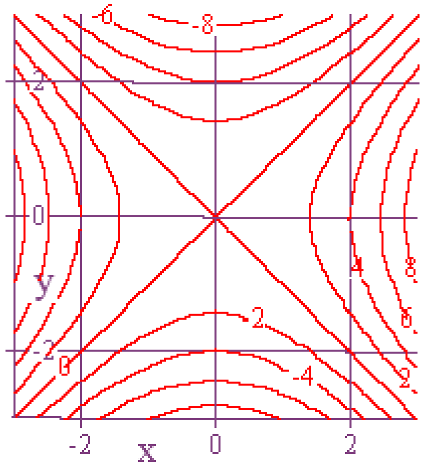


Almost all values of *z* give us hyperbolas. The only exception is if we set $z = 0$. In this case, we get $0 = x^2 - y^2 \Rightarrow y^2 = x^2 \Rightarrow y = \pm x$. When we graph the last equation, we get two straight lines that intersect at the origin.

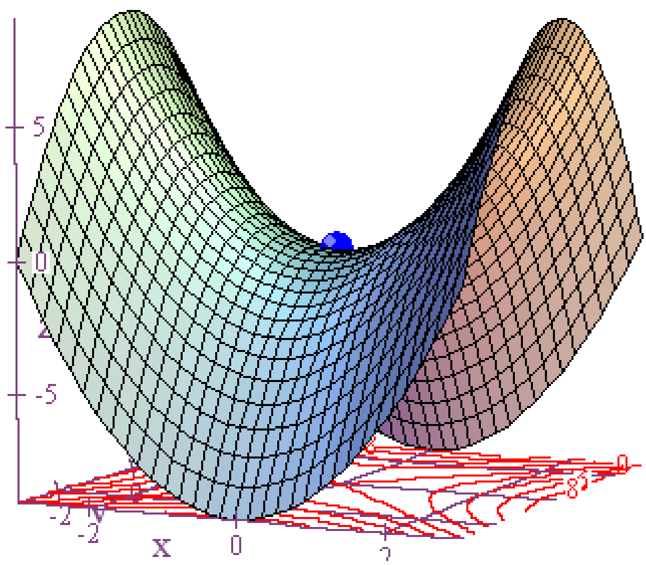


Now let's look at a computer-generated graph along with the level curves.

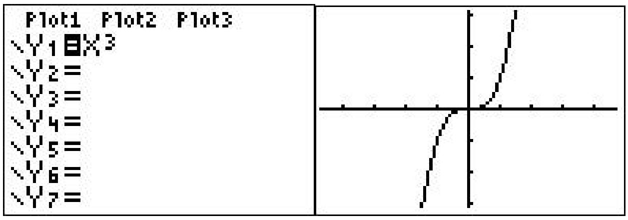




This example is important for two reasons. First, it shows us that we can actually explore level curves on a standard graphing calculator by setting *z* to a fixed value and solving the equation for *y* in terms of *x*. And second, notice the point below at coordinates $(0,0,0)$.

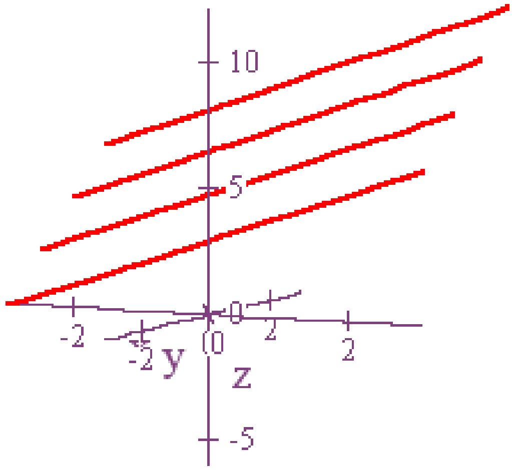


The surface seems to level off momentarily at this point, but if we then proceed in one direction, the values of *z* increase while if we go in another direction they decrease. Because all three of these things happen, (1) leveling off, (2) increasing in one direction, and (3) decreasing in another, and because of the shape of the graph itself, we call the point $(0,0,0)$ a saddle point. This concept is important and will be explored in more detail later on. For now, however, just recall that one of your main goals in Calculus I was to find maximum and minimum points. These points generally occur at places where the graph momentarily levels off, or in more technical language, where the slope of the tangent line becomes zero. Similarly, in higher dimensions, maximum and minimum points will also tend to happen where things level off, but sometimes we will wind up with saddle points, instead. A lower dimensional version of a saddle point could be something like the point $(0,0)$ on the graph of $y = x^3$. Here, once again, we might say there is leveling off, but no cigar. No maximum or minimum.

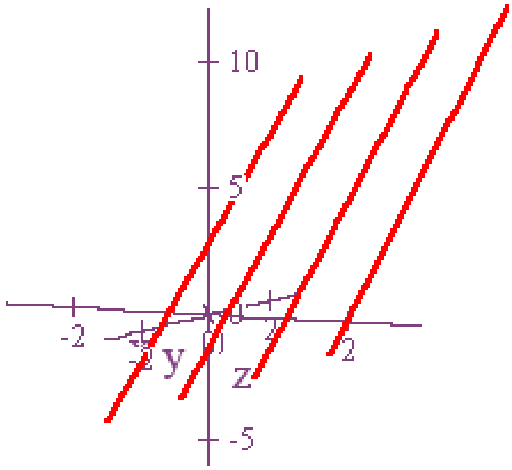


Example 2: $z = x + 2y + 3$

Let's think for a moment about what's going to happen with this function. If we fix a value of y, then we get z as a linear function of x. For example, set $y = 0,1,2,\&3$, in turn, and we get back the linear equations $z = x + 3$, $z = x + 5$, $z = x + 7$, and $z = x + 9$. All of these equations represent lines of slope 1, and if we graph them in the appropriate locations in 3-dimensional space, then we start to see that the graph of $z = x + 2y + 3$ is a plane.



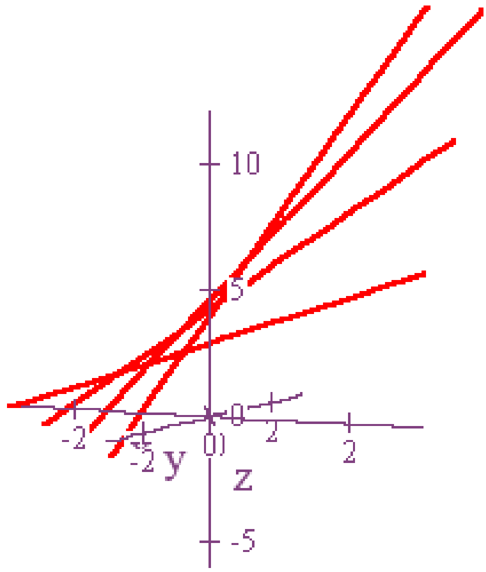
If instead of fixing $y = 0,1,2,\&3$, we fix x to the same, then we'll get the equations $z = 2y + 3$, $z = 2y + 4$, $z = 2y + 5$, and $z = 2y + 6$. The graphs of these lines all have slope 2, and they, likewise, describe a plane.



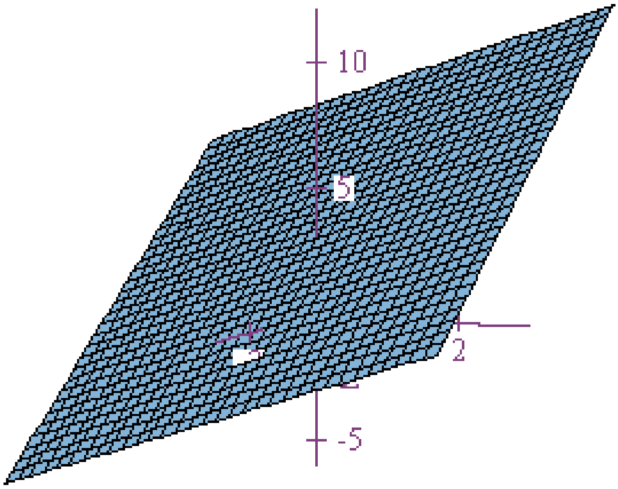
A few things should be clear at this point. Namely,

1. Cross-sections of a plane obtained by fixing a value of *y* will be straight lines, and these lines will all have the same slope.
2. Cross-sections of a plane obtained by fixing a value of *x* will be straight lines, and these lines will all have the same slope.
3. The slope of the linear cross-sections obtained by fixing a value of *y* represents the rate at which *z* changes with respect to *x* as we move in the direction of the positive *x*-axis.
4. The slope of the linear cross-sections obtained by fixing a value of *x* represents the rate at which *z* changes with respect to *y* as we move in the direction of the positive *y*-axis.

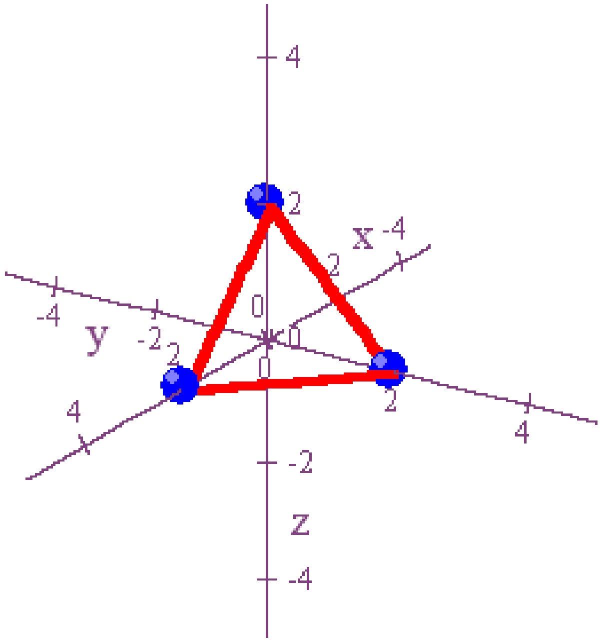
To see why the linear cross-sections of a plane obtained by fixing either *x* or by fixing *y* always have to have the same slope, just look at what happens when the lines don't have the same slope. It's clear from the diagram below that when the cross-sections have different slopes, they are not going to describe a plane.



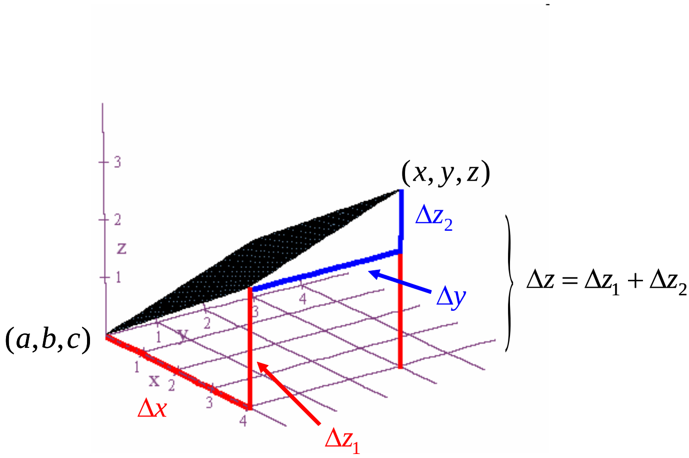
If we now look at the complete graph of $z = x + 2y + 3$, we do indeed see that the result is a plane.



If we are trying to graph a plane such as $z = x + 2y + 3$ by hand, then the easiest way to do it is often by plotting the *x*-, *y*-, and *z*-intercepts. As long as these three points are distinct, this method works well. Below is what we get for the plane $x + y + z = 2$, and simply connecting these three dots is enough to give us an idea of what the final plane looks like.

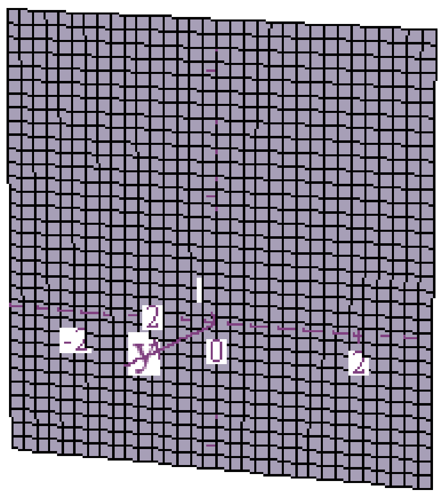


Now let's turn our attention, for a moment, back to the algebraic equation for the plane, $z = x + 2y + 3$. One thing we notice is that each term is either a constant term, or the term contains a single variable raised to only first power. This should remind you of the form of a linear equation in two variables, $y = mx + b$, and it makes sense that we should be seeing something similar in 3-dimensions since a plane is simply a higher dimensional version of a line. What we would like to do at this point is to convince ourselves that the graph of something of the form $z = Ax + By + C$ is always going to be a plane. To do this, we will utilize a diagram that we'll use with variations several times throughout this book. So, in other words, remember what we do here!

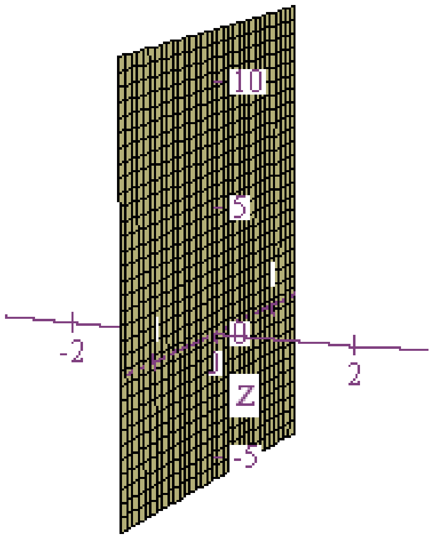


In this diagram, we show a plane that contains the point (a,b,c) and the point (x,y,z). As we move from the first point to the second, there is a change in *x*, a change in *y*, and a change in *z*. We denote these, respectively, by $\Delta x = x - a$, $\Delta y = y - b$, and $\Delta z = z - c$. Notice also that the slope of a cross-section in the direction of the positive *x*-axis is $\frac{\Delta z_1}{\Delta x}$, the slope of a cross-section in the direction of the positive *y*-axis is $\frac{\Delta z_2}{\Delta y}$, and $\Delta z = \Delta z_1 + \Delta z_2$. If we call the slope in the direction of the positive *x*-axis m_x and the slope in the direction of the positive *y*-axis m_y, then $m_x = \frac{\Delta z_1}{\Delta x}$ and $m_y = \frac{\Delta z_2}{\Delta y}$ imply that $\Delta z_1 = m_x \Delta x$ and $\Delta z_2 = m_y \Delta y$. From this it follows that $\Delta z = \Delta z_1 + \Delta z_2 = m_x \Delta x + m_y \Delta y$. If we now replace the changes in *x*, *y*, and *z* by the expressions we have above, then we get $z - c = m_x (x - a) + m_y (y - b)$ which could be written in a more general form as $z = Ax + By + C$. In this form, *A* is the slope of the linear cross-section as one moves in the direction of the positive *x*-axis, and *B* is the slope in the direction of the positive *y*-axis. Notice, too, that $Az + Bx + Cy + D = 0$ is just a variation of the above form, and it also represents a plane. If $A \neq 0$, then we can rewrite this as

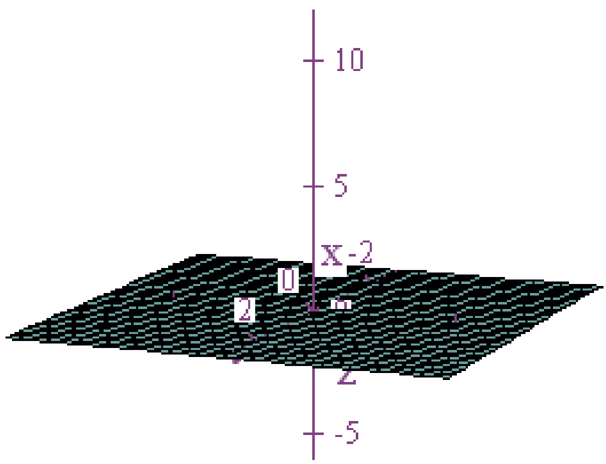
$z = \frac{-B}{A}x + \frac{-C}{A}y + \frac{-D}{A}$ which is equivalent to the general form we wrote above. Particular special cases occur when some of the coefficients in $Az + Bx + Cy + D = 0$ are equal to zero. For example, the graph of $x = 0$ is just the *yz*-plane.



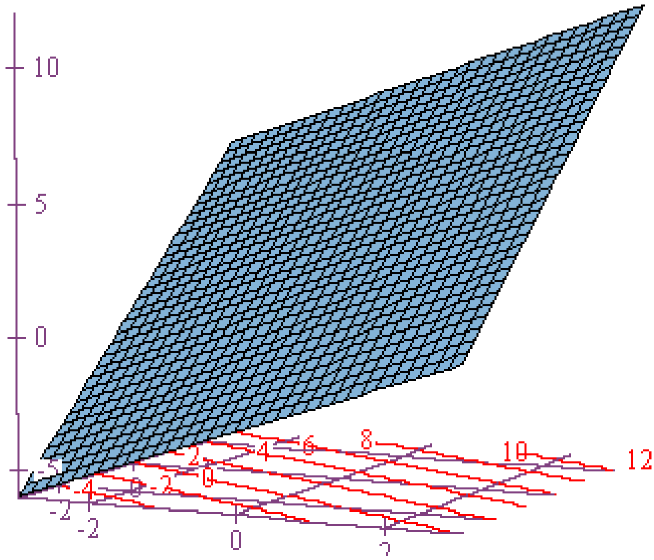
The graph of $y = 0$ is the *xz*-plane.

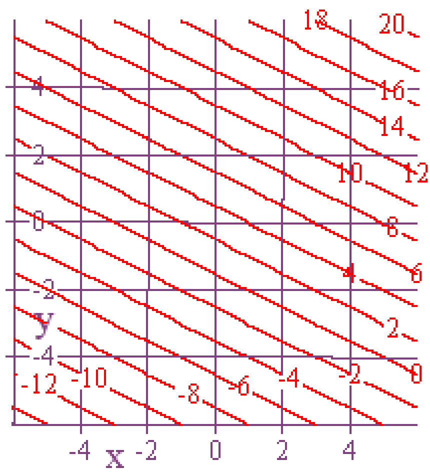


And the graph of $z = 0$ is the *xy*-plane.



If we return for a moment to the function we started this example with, $z = x + 2y + 3$, below is the graph of this plane along with its level curves.

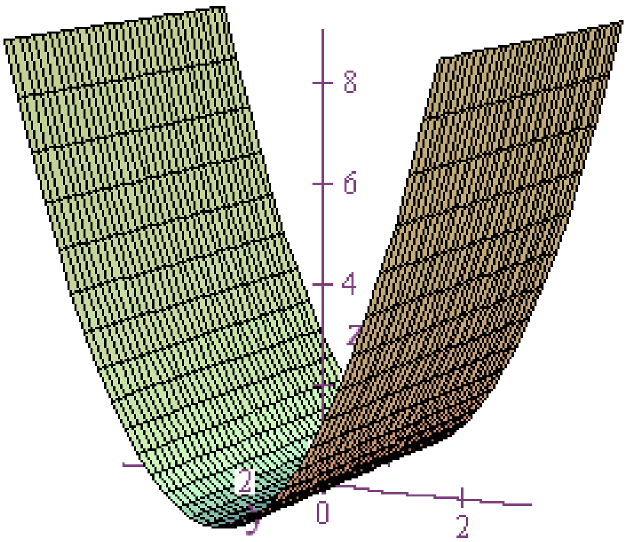




Notice that not only are the level curves straight lines, they also appear to be evenly spaced. Level curves are usually constructed so that the change in z is always the same as we go from one curve to the next, and when the curves are also evenly spaced as we move in either the positive x or y direction, then this shows that the values for z are changing at a constant rate with respect to both x and y. This is exactly what we expect to happen since if we fix either the x or the y value, then the cross-section is a straight line, and straight lines have a constant rate of change. In the next example, though, we'll see what happens when the level curves are straight lines, but the rate of change of z with respect to one of the other variables is not constant.

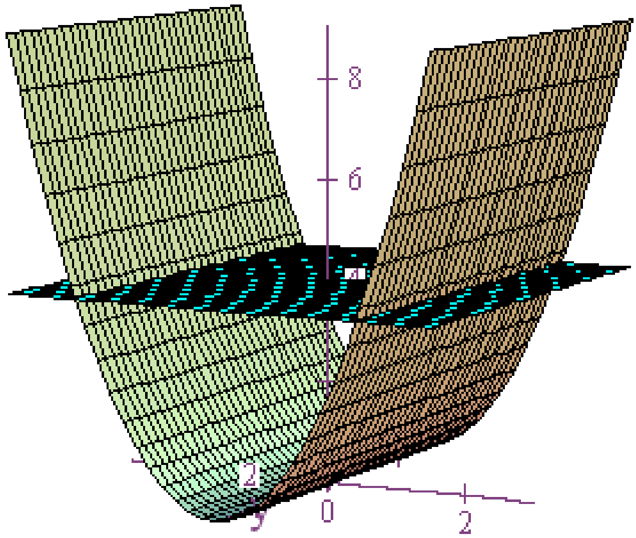
Example 3: $z = y^2$

In this function there is no x variable. What this means is simply that the value of x doesn't matter. In other words, the value of z is completely determined by y. As a result, if we take cross-sections by fixing values for x, then we'll always get the same parabolic curve as our cross-section. Below is the graph of $z = y^2$.

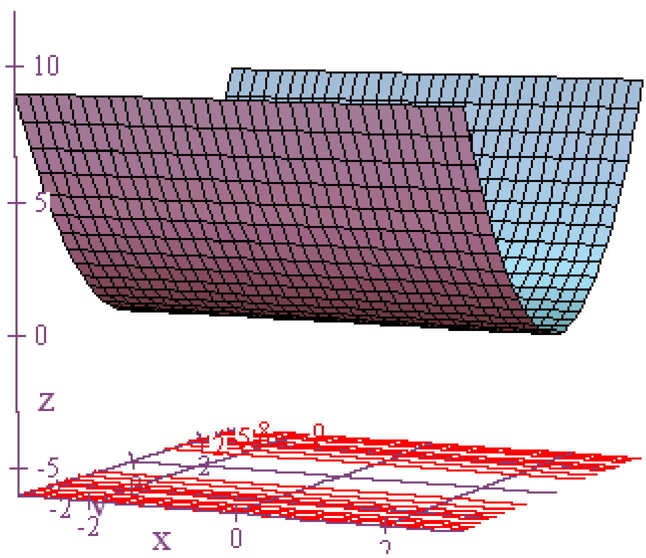


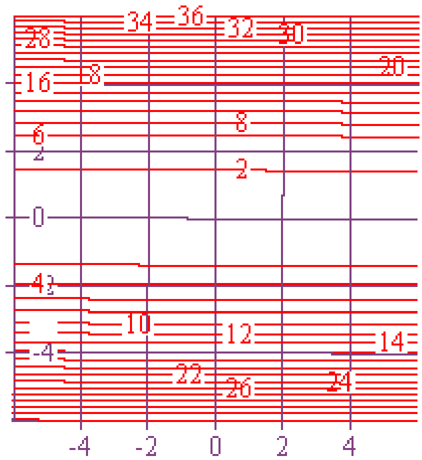
As you can see, the two dimensional parabolic curve defined by $z = y^2$ is just extended in both directions along the *x*-axis. In a situation like this where one of the input variables is completely absent, we call the resulting graph a *cylinder*, and this particular graph is known as a *parabolic cylinder*.

Now let's consider the level curves that result from taking cross-sections of our surface with planes of the form $z = c$. Below is the graph of the intersection of our surface with $z = 4$.



We can see quite easily that the cross-section is a pair of lines. If we now look at a more complete graph of the level curves, we get something like the diagrams below.



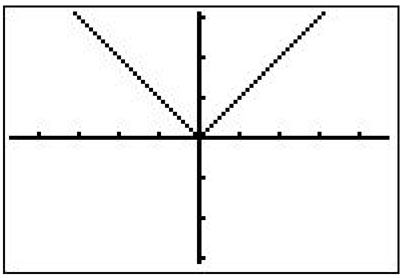


Notice that even though the level curves are all straight lines, they are getting closer together as the y-values on the vertical axis get further away from zero. This is telling us that as we mover further away from the origin on the y-axis, it takes a smaller and smaller increase in y in order to produce the same increase in z. In other words, the z value increases faster and faster as we move away from the origin along the axis of y, and this change in the rate of increase is what immediately tells us that these are not level curves for a plane.

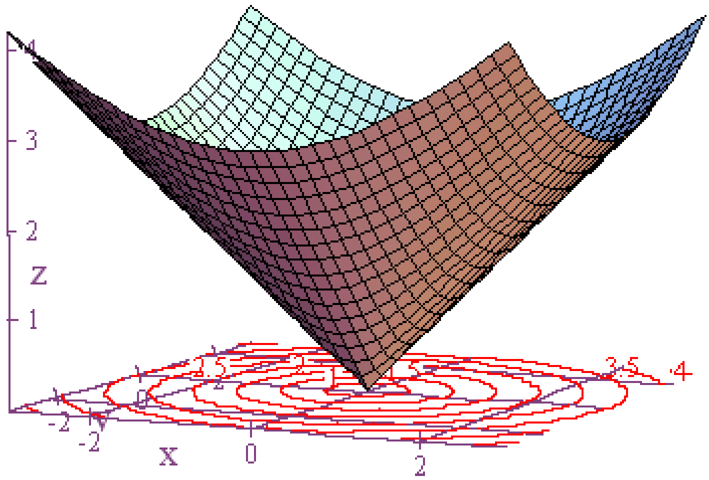
Example 4: $z = \sqrt{x^2 + y^2}$

Sometimes you have a function where the input variables, x and y, always appear together as the expression $x^2 + y^2$. In a situation like this, it's important to remember that $x^2 + y^2 = r^2$ is an equation for a circle of radius r with center at the origin. In other words, if we create a level curve by setting, for example, z equal to 2, then the graph of $x^2 + y^2 = 2^2$ is a circle of radius 2 with center at the origin. More generally, every level curve of $z=\sqrt{x^2+y^2}$ is a circle with radius *z*.

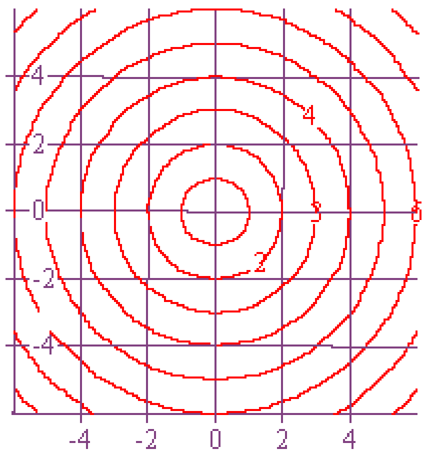
Now let's go a little further with our analysis. If we set $y=0$, then our function becomes $z=\sqrt{x^2}=|x|$. The graph of this curve in two dimensions has the shape of the letter V where the acute angle between each ray and the *x*-axis is $45°$ or $\frac{\pi}{4}$ radians. Below is what the graph looks like on a TI-84 caclculator.



If we set $x=0$, then we get $z=\sqrt{y^2}=|y|$, and the resulting graph of the cross-section in two dimensions looks the same. Now let's think about what happens if we take our point (x,y) off of a circle of radius *r* with center at the origin. In this case we have $z=\sqrt{x^2+y^2}=\sqrt{r^2}=r$. In other words, if you move a straight line distance *r* away from the origin, then you have to move up the same number of units to get to the corresponding point on your surface. The ultimate implication of this is that the graph of $z=\sqrt{x^2+y^2}$ is a cone whose sides in every direction always make a $45°$ angle with the *xy*-plane. Now let's take a look at some graphs for confirmation. I like to call this graph an ice cream cone with no ice cream in it!



4
3
2
z
1
-2
-2
y
x
0
2
3.5
4



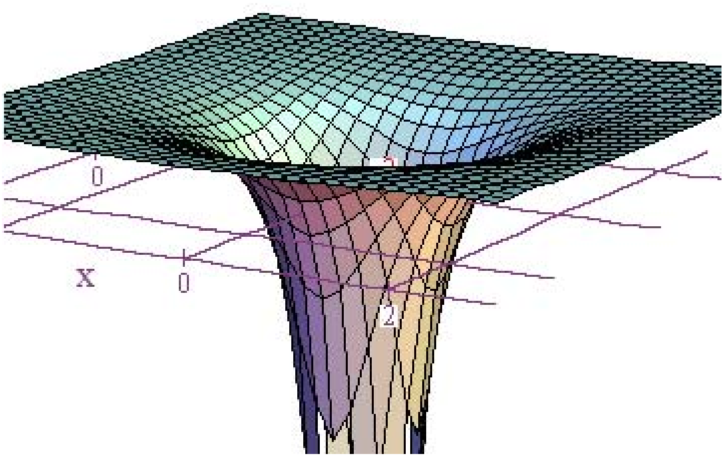
4
2
0
-2
-4
-4
-2
0
2
4

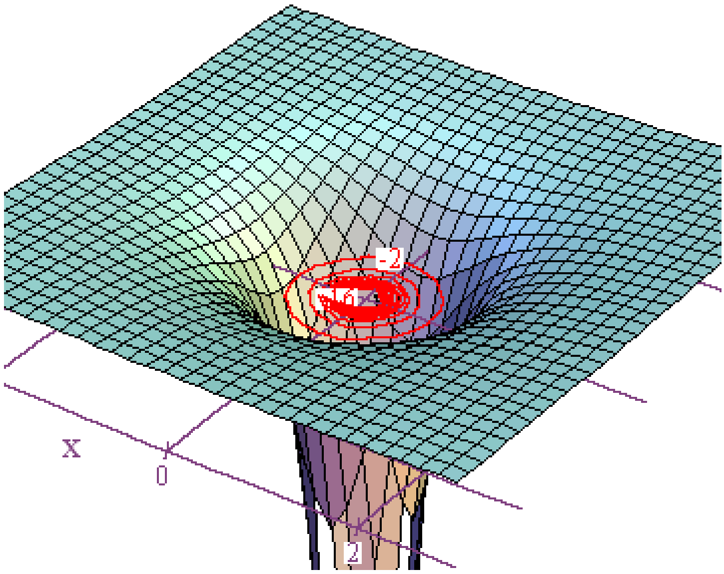
Example 5: $z = \dfrac{-1}{x^2 + y^2}$

Like the previous function, this one also has the input variables bundled together in the expression $x^2 + y^2$. This means that the level curves obtained by fixing values of *z* will also be circles. Additionally, we can also make the following observations.

1. The values for *z* are always negative.
2. The function is undefined at the origin.
3. The *z* values get larger in magnitude in the negative direction as both *x* and *y* get closer to 0.
4. The *z* values get closer to 0 as either *x* or *y* gets further from the origin.

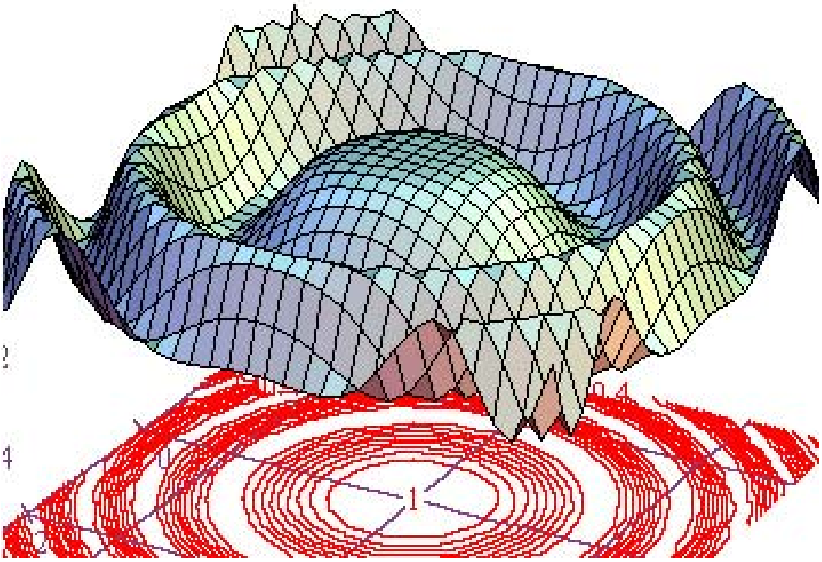
The end result is a graph that looks like the proverbial black hole as it is sometimes depicted graphically in movies.

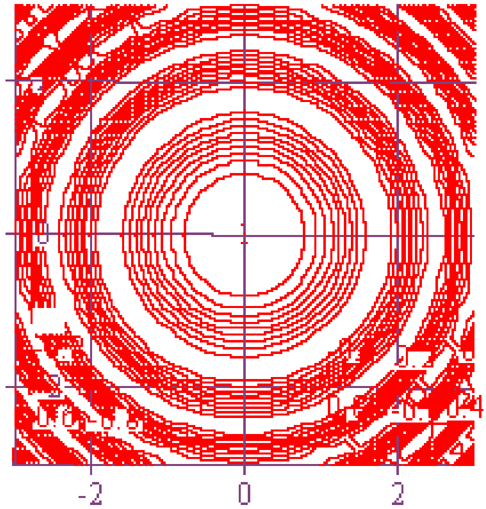




Example 6: $z = \cos(x^2 + y^2)$

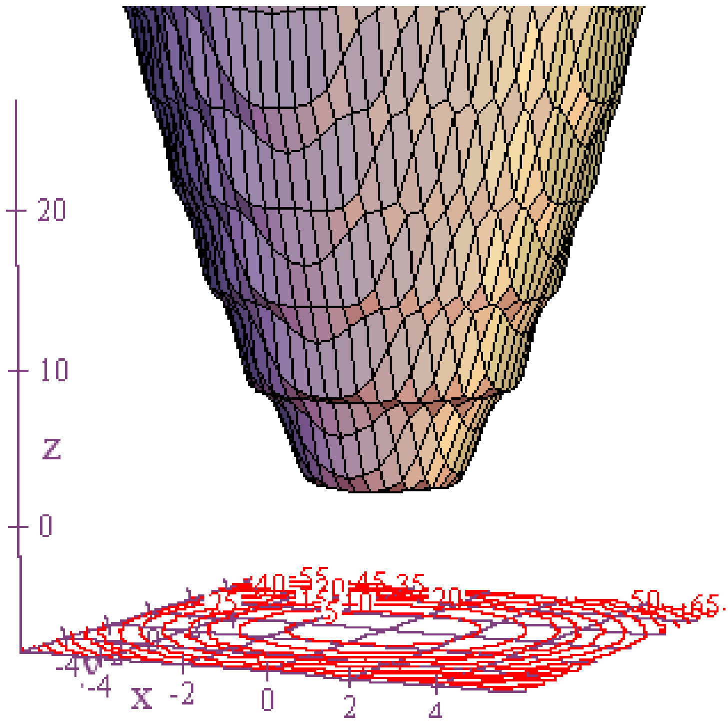
Once again we see the expression $x^2 + y^2$, and we know that the level curves will be circles, or in this case, a horizontal plane can intersect the surface in many concentric circles since the cosine function is periodic with period 2π. Let's now think for a moment, though, about what's going to happen as both *x* and *y* move away from the origin. When the input values get far away from the origin, we usually refer to what happens as the end or long run behavior of the graph. While you might think that such things are complicated, in practice only a few things generally happen. With most functions the long run behavior is generally either that *z* gets large in the positive direction, *z* gets large in the negative direction, *z* becomes asymptotic to some fixed value, or *z* oscillates or jumps around between values. For our function above, it's going to be the latter that happens. For example, since *z* is defined as the cosine of something, we know immediately that the output is going to oscillate between -1 and 1, and , as mentioned, a given output value will often correspond to several circles in the *xy*-plane. For example, the circles $x^2 + y^2 = 2\pi$, $x^2 + y^2 = 4\pi$, and $x^2 + y^2 = 6\pi$ will all be associated with the output value $z = 1$. Let's now look at the surface and its level curves. As deduced above, the level curve diagram consists of concentric circles.

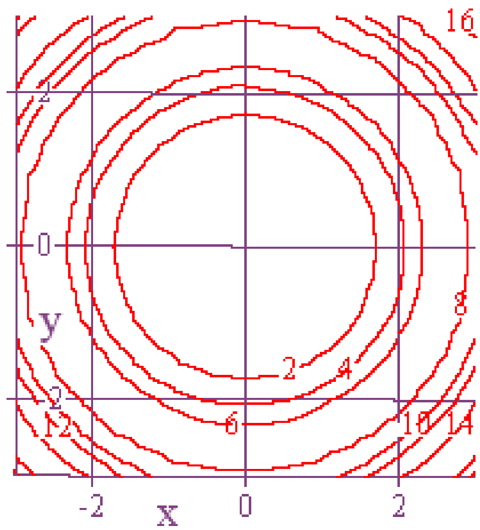




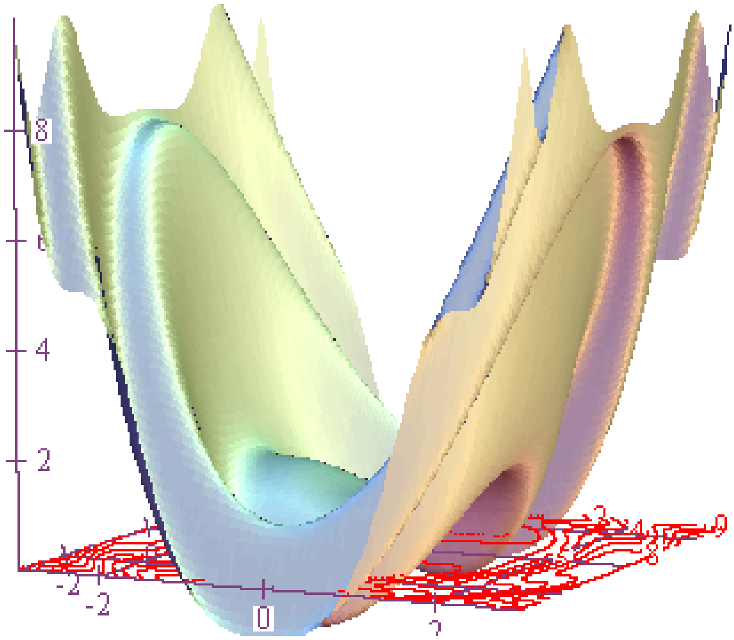
Now let's think about what happens if we change our function to

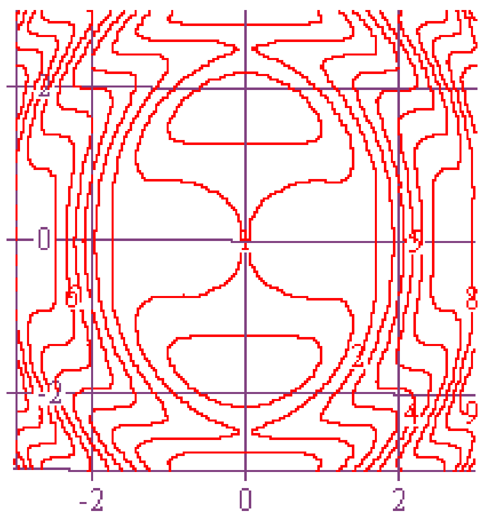
$z = \cos(x^2 + y^2) + (x^2 + y^2)$. This time, the *z* values should increase as *x* and *y* move further from the origin, but the level curves should still be circles since we never find x^2 without a y^2 being added to it. Here is the graph. As we move away from the origin, the *z* values increase, but our cosine term, nevertheless, still creates some oscillation in the output.


20
10
z
0
x
-4
-2
0
2
4

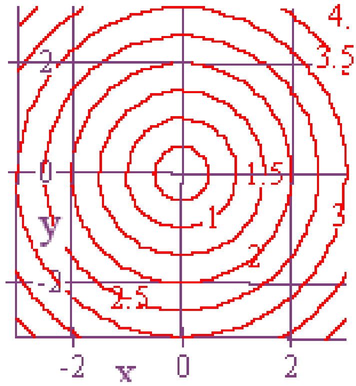

16
2
0
y
-2
8
2
4
6
10
12
14
-2
x
0
2

And finally, let's look at $z = \cos(x^2 + y^2) + x^2$. The graph of $z = x^2$ would be a parabolic cylinder, but once again the cosine term creates some oscillations in the output. Also, since the final x^2 term isn't paired by addition with a y^2, we shouldn't expect to get circles again for the level curves. In fact, what we do get looks a little weird. Welcome to my world!

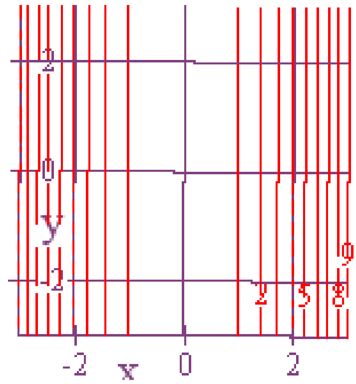




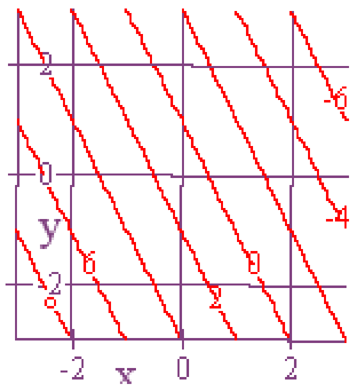
Now let’s have a little fun! Below are four level curve diagrams, but only one of them corresponds to the function $z = -2x + y + 1$. Can you tell which one it is?



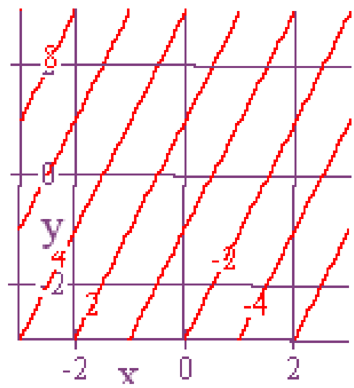
a.



b.



c.



d.

Since $z = -2x + y + 1$ is an equation for a plane, the level curves have to be straight lines and so the answer can’t be *a*. Also, since *z* has to have a constant rate of change with respect to both the positive *x* direction and the positive *y* direction, the answer can’t be *b*. In *b*, the level curves are getting closer together as we move further away from zero along the *x*-axis, and this means that the value of *z* is changing at an ever faster rate. Having eliminated both *a* and *b*, that leaves only *c* and *d*, and now things are trickier because both of these level curve diagrams do look like they represent planes. However, what we need to remember here is that in the equation $z = -2x + y + 1$, the coefficient of *x* tells us the rate at which *z* is changing with respect to movement in the direction of the positive *x*-axis, and the coefficient of *y* tells us the rate at which *z* is changing with respect to movement in the direction of the positive *y*-axis. What we see from these coefficients is that *z* should decrease as *x* increases, and *z* should increase as *y* increases. The only diagram for which this happens is *d*, and so *d* is the correct level curve diagram for $z = -2x + y + 1$.

Example 7: Another way in which we can study level curves for a function of two variables is by using a table of values for our function. For example, below is a table of values for the function $z = x^2 + y^2$, and I've colored the cells corresponding to $z = 1, 5, 13, \& \ 25$. We can already see the circular cross-sections appearing, and we can use this diagram to finish drawing in the corresponding level curves.

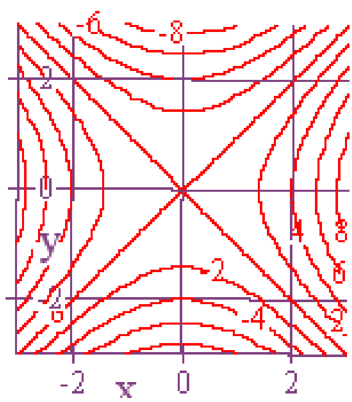
x\y	-5	-4	-3	-2	-1	0	1	2	3	4	5
-5	50	41	34	29	26	25	26	29	34	41	50
-4	41	32	25	20	17	16	17	20	25	32	41
-3	34	25	18	13	10	9	10	13	18	25	34
-2	29	20	13	8	5	4	5	8	13	20	29
-1	26	17	10	5	2	1	2	5	10	17	26
0	25	16	9	4	1	0	1	4	9	16	25
1	26	17	10	5	2	1	2	5	10	17	26
2	29	20	13	8	5	4	5	8	13	20	29
3	34	25	18	13	10	9	10	13	18	25	34
4	41	32	25	20	17	16	17	20	25	32	41
5	50	41	34	29	26	25	26	29	34	41	50

x\y	-5	-4	-3	-2	-1	0	1	2	3	4	5
-5	50	41	34	29	26	25	26	29	34	41	50
-4	41	32	25	20	17	16	17	20	25	32	41
-3	34	25	18	13	10	9	10	13	18	25	34
-2	29	20	13	8	5	4	5	8	13	20	29
-1	26	17	10	5	2	1	2	5	10	17	26
0	25	16	9	4	1	0	1	4	9	16	25
1	26	17	10	5	2	1	2	5	10	17	26
2	29	20	13	8	5	4	5	8	13	20	29
3	34	25	18	13	10	9	10	13	18	25	34
4	41	32	25	20	17	16	17	20	25	32	41
5	50	41	34	29	26	25	26	29	34	41	50

By following the same procedure for the function $z = x^2 - y^2$, we can use this function's table of values to help us replicate the lines and hyperbolas found in its level curve diagram.

x\y	-5	-4	-3	-2	-1	0	1	2	3	4	5
-5	0	9	16	21	24	25	24	21	16	9	0
-4	-9	0	7	12	15	16	15	12	7	0	-9
-3	-16	-7	0	5	8	9	8	5	0	-7	-16
-2	-21	-12	-5	0	3	4	3	0	-5	-12	-21
-1	-24	-15	-8	-3	0	1	0	-3	-8	-15	-24
0	-25	-16	-9	-4	-1	0	-1	-4	-9	-16	-25
1	-24	-15	-8	-3	0	1	0	-3	-8	-15	-24
2	-21	-12	-5	0	3	4	3	0	-5	-12	-21
3	-16	-7	0	5	8	9	8	5	0	-7	-16
4	-9	0	7	12	15	16	15	12	7	0	-9
5	0	9	16	21	24	25	24	21	16	9	0

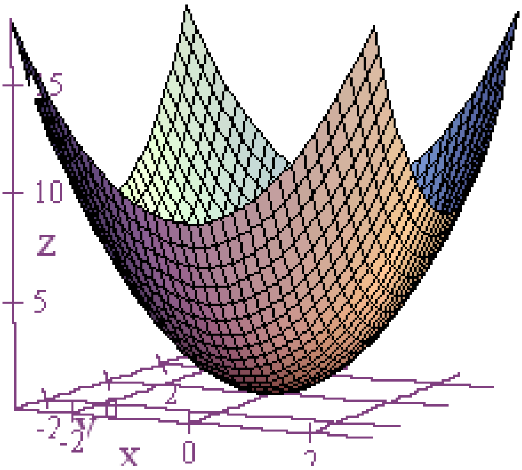
And here is what the corresponding level curve diagram should look like.



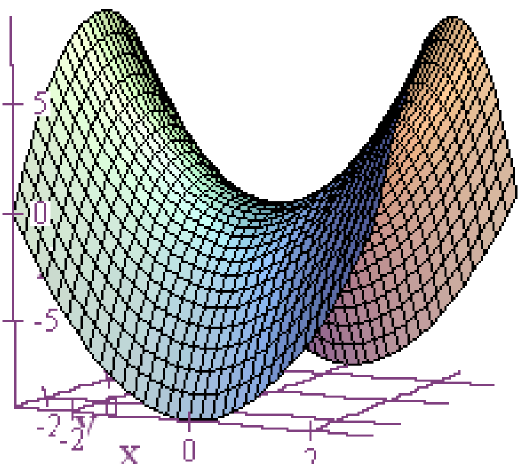
Up to this point, pretty much all of the functions we've looked at have been abstract, mathematical functions of two variables, and for a while longer we will continue to look at things from a pure mathematical perspective. Nonetheless, keep in mind that everything that we are doing can also serve as a mathematical model for real world situations. For example, in a real world context, our input variables might represent levels of production in a factory for different products, and the output variable could be the corresponding profit generated. Or the input variables could represent a person's height and weight, and the output could be the probability of a person having approximately that height and weight. We'll see a variety of real world problems later on, and for those problems the units attached to the variables will provide a practical context for our answers. For now, however, we will continue studying things at a purely abstract level.

For the final set of examples, I just want to present a catalog of some of my favorite graphs of functions of two variables. Each is a basic example in its own way, and several of them we've seen before. Nevertheless, study them well, contemplate why the graphs look the way they do, and enjoy!

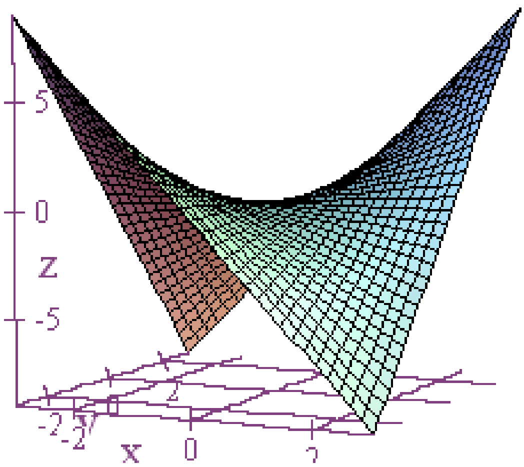
1. $z = x^2 + y^2$



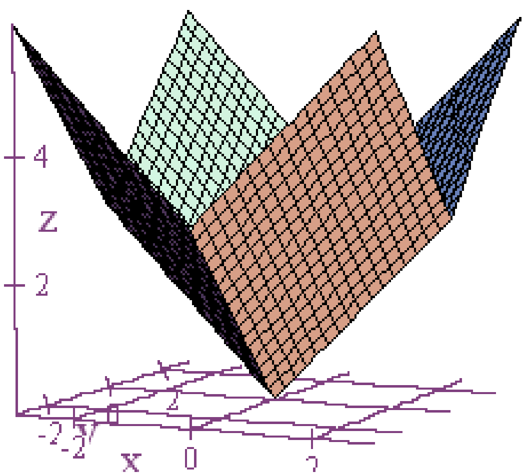
2. $z = x^2 - y^2$



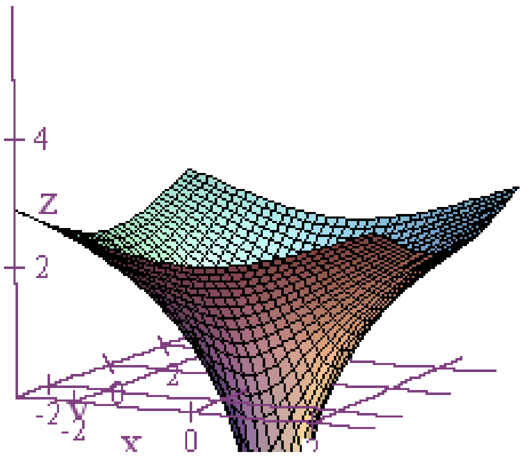
3. $z = xy$



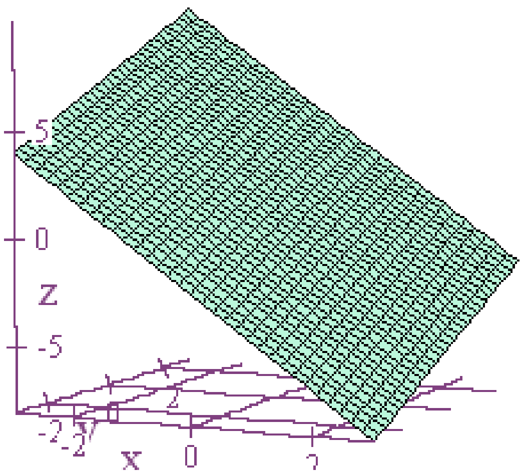
4. $z = |x| + |y|$



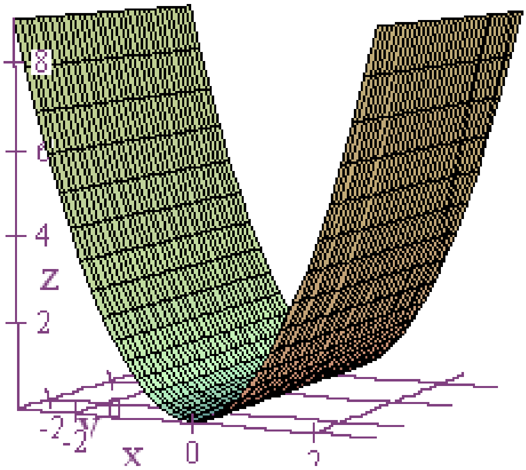
5. $z = \ln\left(x^2 + y^2\right)$



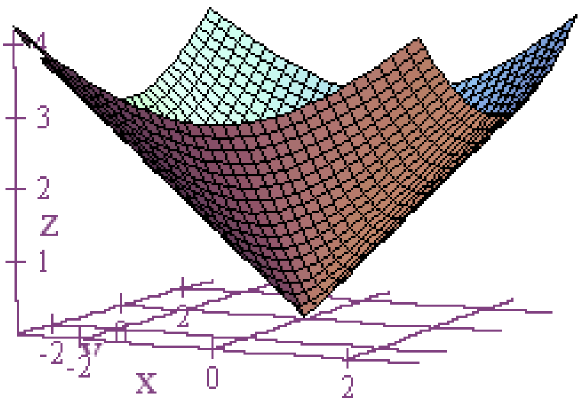
6. $z = -2x + y + 1$



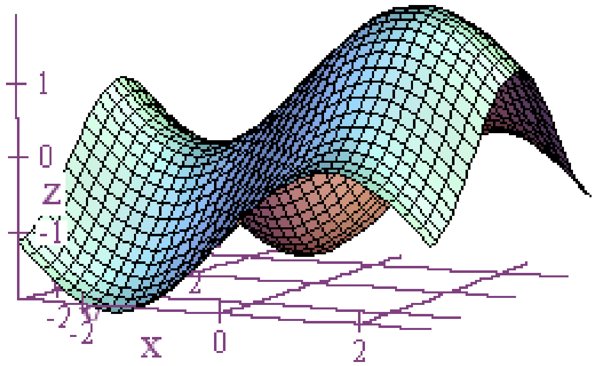
7. $z = x^2$



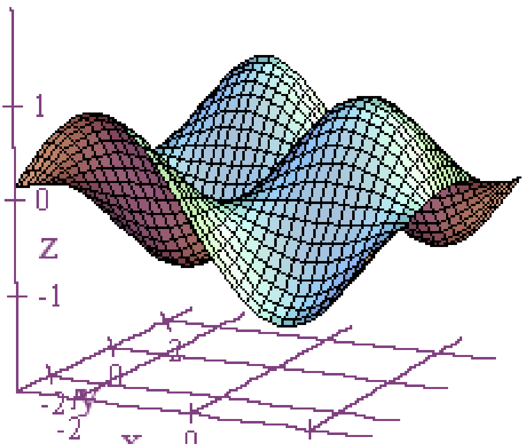
8. $z = \sqrt{x^2 + y^2}$



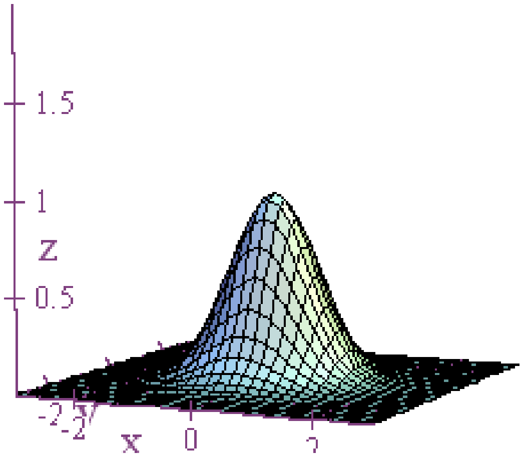
9. $z = \sin(x) + \cos(y)$



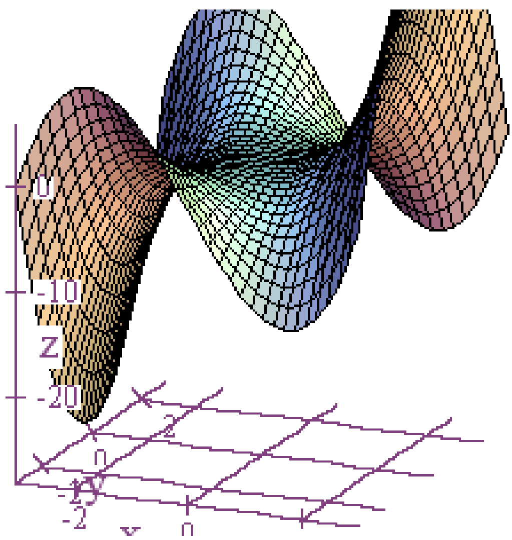
10. $z = \sin(x)\cos(y)$



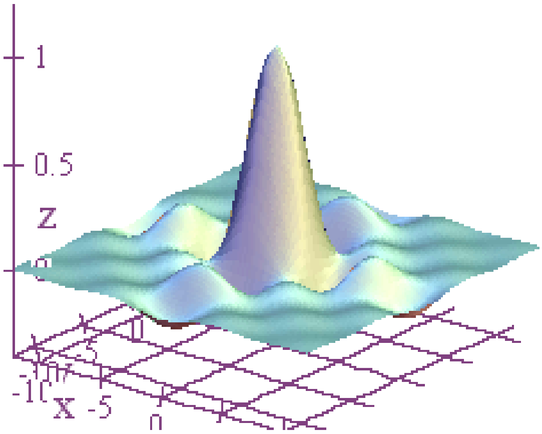
11. $z = e^{-(x^2+y^2)}$



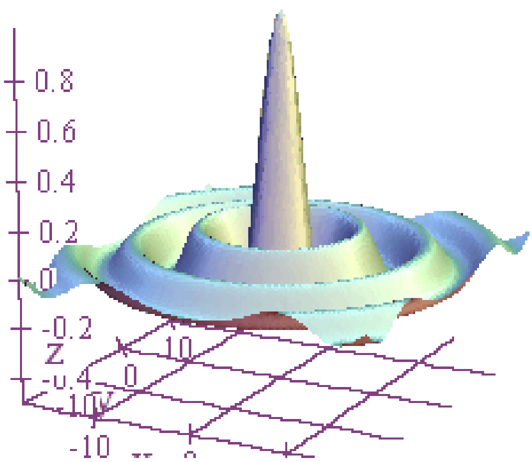
12. $z = x^3 - xy^2$



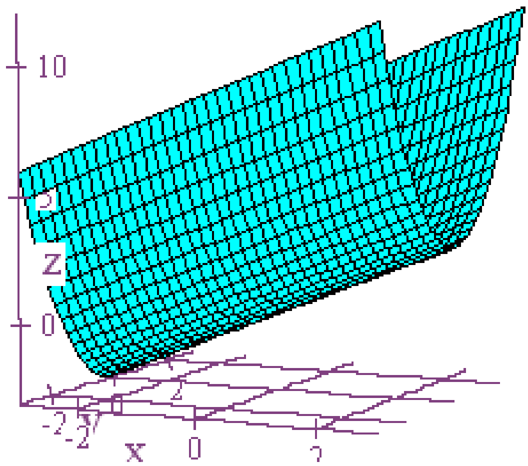
13. $z = \dfrac{\sin(x)\sin(y)}{xy}$



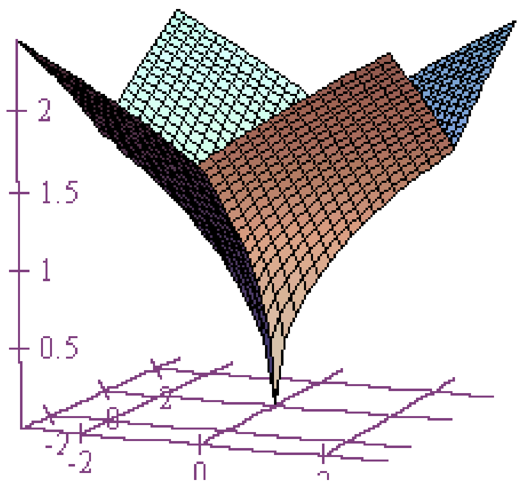
14. $z = \dfrac{\sin\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}}$



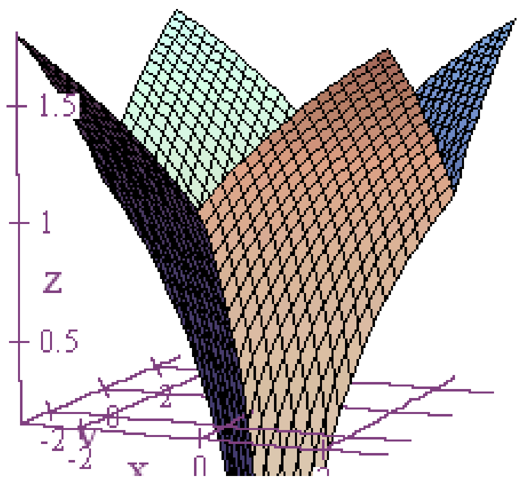
15. $z = y^2 + x$



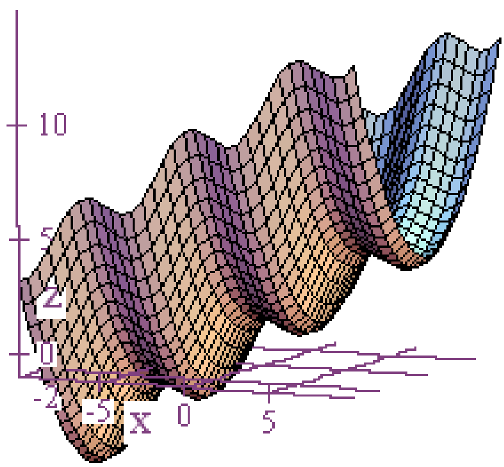
16. $z = \sqrt{|x| + |y|}$



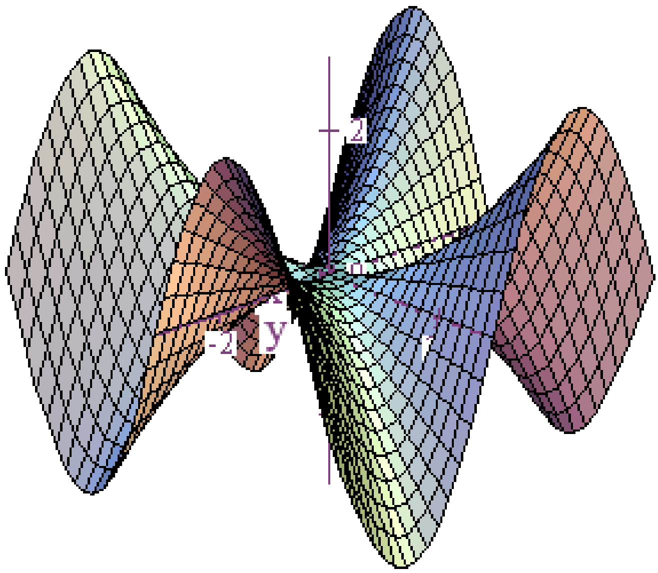
17. $z = \ln(|x| + |y|)$



18. $z = y^2 + \cos(x) + \dfrac{x}{2}$



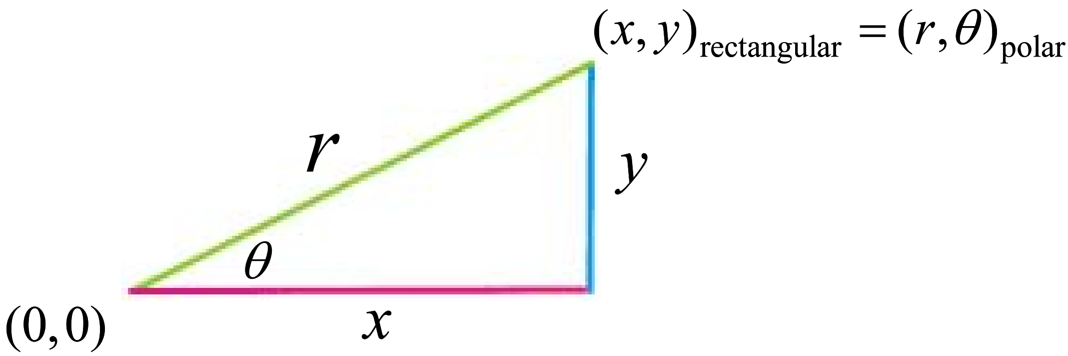
19. $z = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$



CHAPTER 2

CYLINDRICAL AND SPHERICAL COORDINATES

Okay! You've finished chapter 1, studied graphs of functions of several variables in exhaustive detail, and now you know everything you need to know about graphs in 3-dimensions. Right? Wrong!!! Remember polar coordinates? Well, we hope you remember polar coordinates. As you hopefully recall, polar coordinates are another way to locate points in 2-dimensional space, and they serve as an alternative to the usual rectangular coordinates that locate a point using values off of the x and y axes. In polar coordinates, instead of using x and y to locate a point, we specify the point's distance from the x-axis and the angle that it makes with the positive x-axis. Here's the diagram that explains it all.

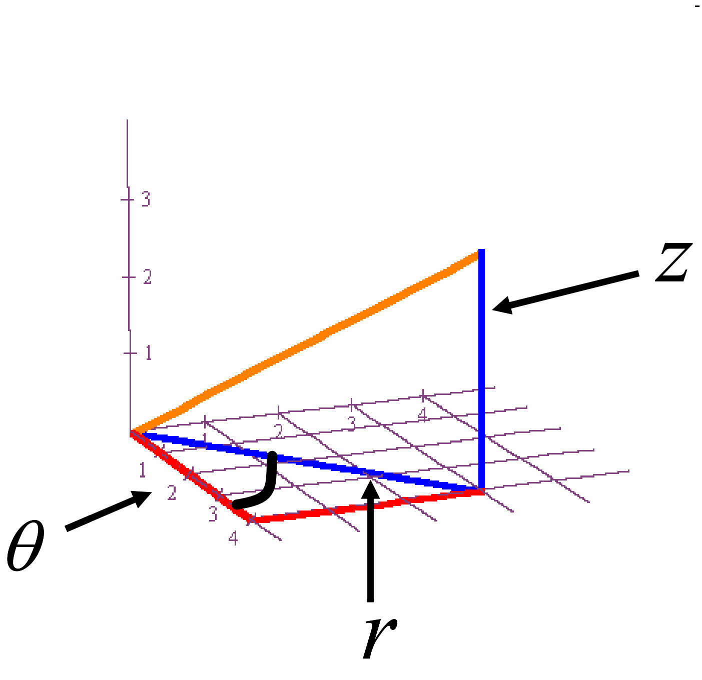


From trigonometry, we also know that the following relationships will exist between rectangular coordinates (x, y) and polar coordinates (r,θ):

$x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$, and $slope = \tan\theta = y/x$.

The cylindrical and spherical coordinate systems that we are going to look at now might be thought of as extensions of polar coordinates into higher dimensions. We'll begin with cylindrical coordinates.

In cylindrical coordinates for a point in 3-dimensional space, we basically make the first two coordinates polar and then keep the elevation z that we are above or below the xy-plane as our third coordinate. Here's an illustration that shows how we can locate a point in space with coordinates (r, θ, z).



The position of the point as it might lie in the xy-plane is determined by the polar coordinates r and θ, and then the distance above or below this plane is determined by the z coordinate. The important relationships between rectangular coordinates, (x, y, z), and cylindrical coordinates, (r, θ, z), are as follows:

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $x^2 + y^2 = r^2$, and $\tan\theta = y/x$, where we usually also have that $0 \le \theta \le 2\pi$, $0 \le r < \infty$, and $-\infty < z < \infty$.

At his point, there are only a few things we really want to know when it comes to cylindrical coordinates. We want to understand a few basic graphs and when it might be easier to approach a problem using cylindrical rather than rectangular coordinates, and we want to be able to easily convert from one coordinate system to the other. Well start with some conversion examples.

Example 1: Convert $(2, \pi/2, 3)$ from cylindrical, (r, θ, z), to rectangular, (x, y, z), coordinates.

This is the kind of conversion that's easiest to do. We just apply directly the formulas $x = r\cos\theta$ and $y = r\sin\theta$. Thus, $(2, \pi/2, 3)_{\text{cylindrical}}$ is equal to $\left(2\cos(\pi/2), 2\sin(\pi/2), 3\right) = (0, 2, 3)_{\text{rectangular}}$.

Example 2: Convert $(2,\pi/4,1)$ from cylindrical, (r,θ,z), to rectangular, (x,y,z), coordinates.

Again, this is just $\left(2\cos(\pi/4),2\sin(\pi/4),1\right)=\left(2/\sqrt{2},2/\sqrt{2},1\right)=\left(\sqrt{2},\sqrt{2},1\right)_{\text{rectangular}}$.

Example 3: Convert $\left(1,\sqrt{3},4\right)$ from rectangular, (x,y,z), to cylindrical, (r,θ,z), coordinates.

Going from rectangular to cylindrical is only slightly more difficult than going from cylindrical to rectangular. In particular, we will have to use the arctangent function to help us find our angle, and if necessary, we will have to make adjustments to our result to ensure that the angle lies between 0 and 2π. We will also need to calculate the value for r from the values we have for x and y. In this example, $\tan\theta=\frac{y}{x}=\frac{\sqrt{3}}{1}$, and so $\theta=\tan^{-1}\sqrt{3}=60°=\frac{\pi}{3}$. Also, $r=\sqrt{x^2+y^2}=\sqrt{1^2+\left(\sqrt{3}\right)^2}=\sqrt{4}=2$. Therefore, $\left(1,\sqrt{3},4\right)_{\text{rectangular}}=\left(2,\frac{\pi}{3},4\right)_{\text{cylindrical}}$.

<u>Example 4:</u> Convert $\left(-\sqrt{3},1,2\right)$ from rectangular, (x,y,z), to cylindrical, (r,θ,z), coordinates.

This is similar to the previous example except that when we try to find our angle, we have to make an adjustment to get it into the second quadrant where our *x* and *y* coordinates lie. Thus, we know that $\tan\theta = \frac{y}{x} = \frac{1}{-\sqrt{3}}$ and $\tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = -30° = \frac{-\pi}{6}$.

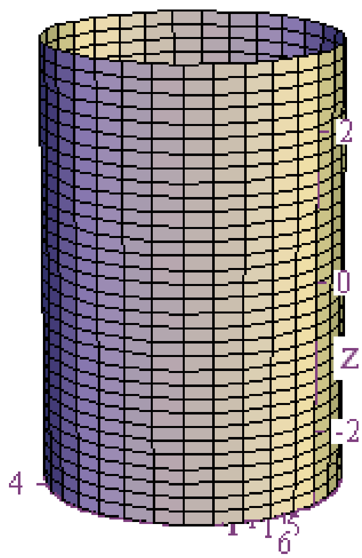
But as you can see, our angle puts us in the wrong quadrant with respect to *x* and *y*, and so we are going to have to find the related second quadrant angle. We can do this by either adding $180°$ to $-30°$ or π radians to $\frac{-\pi}{6}$. We'll chose the latter format since it is more customary to express this angle in radians. Also,

$r = \sqrt{x^2 + y^2} = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$, and therefore,

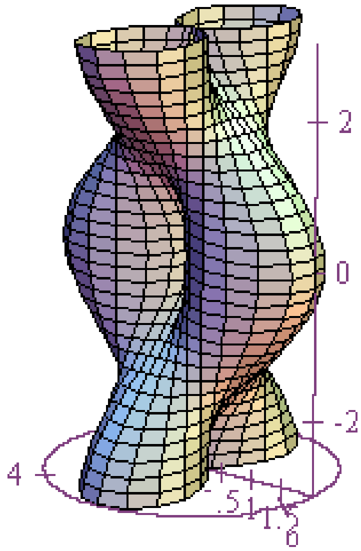
$$\left(-\sqrt{3},1,2\right)_{\text{rectangular}} = \left(2,\frac{5\pi}{6},2\right)_{\text{cylindrical}}.$$

Now let's look at a few graphs in cylindrical coordinates. These are often set up as $r = f(\theta,z)$ where the dependent or output variable is *r* and the two independent or input variables are θ and *z*. It doesn't have to always be set up this way, but nonetheless, that's how it is often done and how we will do it in the few examples below.

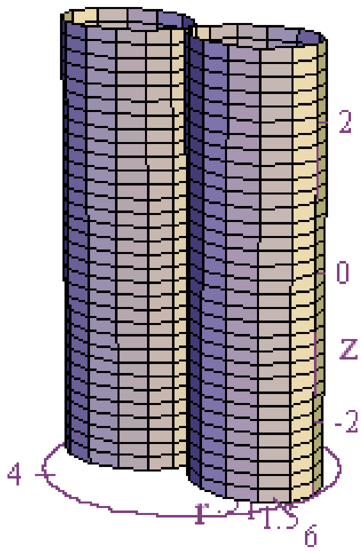
The type of surface that is often described using cylindrical coordinates is one in which the points in the domain are related to either a disk or a circle of radius r. For example, suppose we set $r = 2$, let θ vary so that $0 \le \theta \le 2\pi$, and let z vary so that $-3 \le z \le 3$. Then the result is a cylindrical surface about the z-axis of radius 2 and extending from $z = -3$ to $z = 3$. Because we can describe this surface so easily using cylindrical coordinates, that's where the particular name comes from.



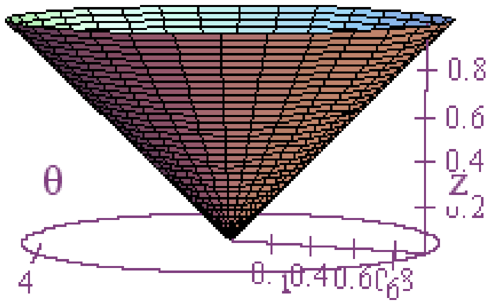
Example 5: There are a variety of interesting surfaces that may be generated using cylindrical coordinates, but they are often more difficult to analyze than surfaces in rectangular coordinates. One of my favorite surfaces, though, is $r = 1 + \cos 2\theta \cos z$ where $0 \le \theta \le 2\pi$ and $-3 \le z \le 3$. I like to think of this graph as two twin souls inextricably joined together.



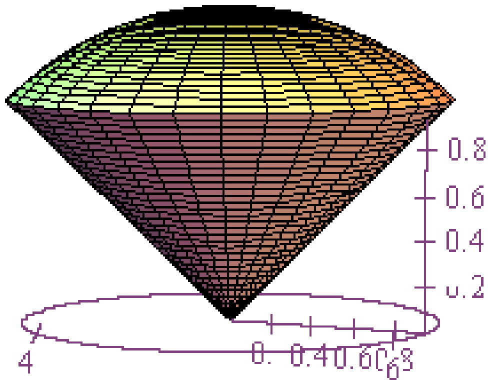
You can see in this graph how multiplying the main term by $\cos z$ creates a nice wave as we ascend up the z-axis. Without this multiplier, the graph $r = 1 + \cos(2\theta)$ looks much more mundane.



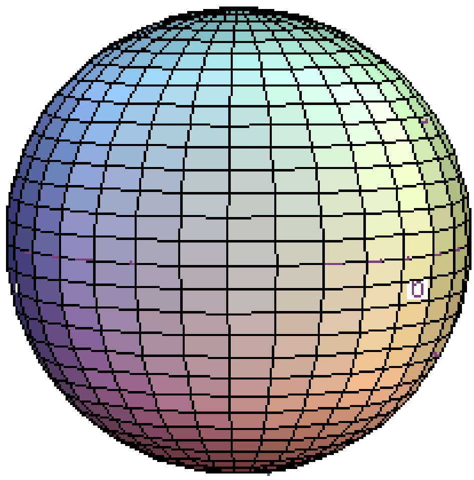
Example 6: We can also use cylindrical coordinates to easily create an ice cream cone and then fill it with ice cream. Recall that we did a representation of a cone earlier by graphing $z = \sqrt{x^2 + y^2}$. If we change this equation to cylindrical coordinates, then it becomes $z = \sqrt{x^2 + y^2} = \sqrt{r^2} = r$. Thus, if we set $0 \le \theta \le 2\pi$, $0 \le r \le 1$, and $z = r$, then we get a cone whose sides have length $\sqrt{2}$.



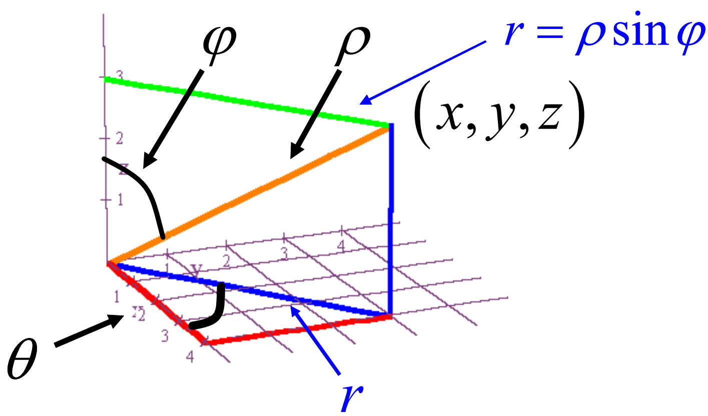
If we want to now fill the ice cream cone with ice cream, then we can generate a portion of a sphere of radius $\sqrt{2}$ for this purpose. The usual equation in rectangular coordinates for a sphere of this radius with center at the origin is $x^2 + y^2 + z^2 = 2$. We can change this to cylindrical coordinates as $z = \sqrt{2-(x^2+y^2)} = \sqrt{2-r^2}$ where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. And this is the result when we combine this surface with our cone.



Now let's start looking at spherical coordinates. To specify a point in spherical coordinates, we need three things – the distance ρ the point is from the origin, the angle θ the point would make with the positive *x*-axis if we repositioned it in the *xy*-plane, and the angle φ that the point makes with the positive *z*-axis. And as you might expect, this is the easiest way to represent a sphere. For the sphere of radius $\sqrt{2}$ the equation simply becomes $\rho = \sqrt{2}$ where $0 \le \theta \le 2\pi$ and $0 \le \varphi \le \pi$. The result is below.



Before we do any more graphs, however, let's look at how we might convert coordinates from one system to the other. Study this diagram well!



From this diagram we can deduce the following relationships:

$x = r\cos(\theta) = \rho\sin(\varphi)\cos(\theta)$, $y = r\sin(\theta) = \rho\sin(\varphi)\sin(\theta)$, and $z = \rho\cos(\varphi)$. We can also conclude that $\rho^2 = x^2 + y^2 + z^2$, $\tan\theta = \frac{y}{x}$, and $\varphi = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{\rho}$.

Furthermore, we generally assume that $0 \le \rho < \infty$, $0 \le \varphi \le \pi$, and $0 \le \theta \le 2\pi$. Now let's look at some examples involving conversions of coordinates.

Example 7: Just as with cylindrical coordinates, it's much easier to convert from spherical coordinates, (ρ,θ,φ), to rectangular coordinates, (x,y,z), than it is the other way around. For example, suppose we want to convert $(1,0,0)_{\text{spherical}}$ to rectangular. We just do the calculations $x = \rho\sin\varphi\cos\theta = 1\cdot\sin 0\cdot\cos 0 = 0$,

$y = \rho\sin\varphi\sin\theta = 1\cdot\sin 0\cdot\sin 0 = 0$, and $z = \rho\cos\varphi = 1\cdot\cos 0 = 1$. Hence,

$(1,0,0)_{\text{spherical}} \to (0,0,1)_{\text{rectangular}}$.

Example 8: Let's do another one! This time we'll convert $(2,\pi/2,3\pi/4)_{\text{spherical}}$ to rectangular coordinates. Once again,

$x = \rho\sin\varphi\cos\theta = 2\cdot\sin 3\pi/4\cdot\cos\pi/2 = 2\cdot\left(\sqrt{2}/2\right)\cdot 0 = 0$,

$y = \rho\sin\varphi\sin\theta = 2\cdot\sin 3\pi/4\cdot\sin\pi/2 = 2\cdot\left(\sqrt{2}/2\right)\cdot 1 = \sqrt{2}$, and

$z = \rho\cos\varphi = 2\cdot\cos 3\pi/4 = 2\cdot\left(-\sqrt{2}/2\right) = -\sqrt{2}$. Hence,

$(2,\pi/2,3\pi/4)_{\text{spherical}} \to (0,\sqrt{2},-\sqrt{2})_{\text{rectangular}}$.

<u>Example 9:</u> Now let's go in the opposite direction and convert $(-3,0,0)_{\text{rectangular}}$ to spherical coordinates. First, $\rho=\sqrt{(-3)^2+0^2+0^2}=3$. Also, if we graph this point, then it's position in the xy-plane with respect to the positive x-axis tells us that $\theta=\pi$, and its position with respect to the positive z-axis tells us that $\varphi=\frac{\pi}{2}$. Therefore,

$$(-3,0,0)_{\text{rectangular}} \to \left(3,\pi,\frac{\pi}{2}\right)_{\text{spherical}}.$$

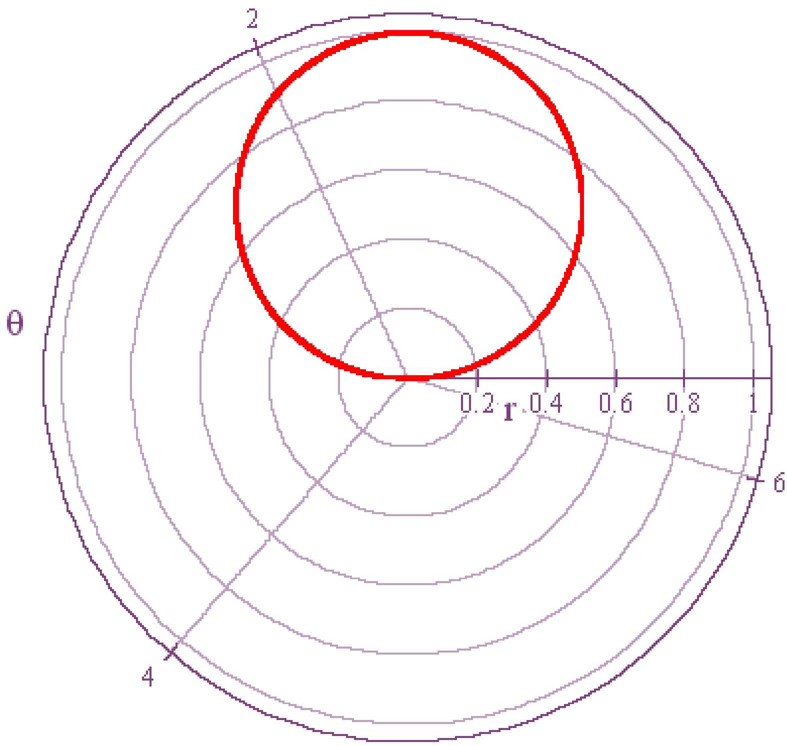
<u>Example 10:</u> For our last conversion, let's convert $(1,-1,-\sqrt{2})_{\text{rectangular}}$ to spherical coordinates. We have $\rho=\sqrt{1^2+(-1)^2+(-\sqrt{2})^2}=\sqrt{4}=2$. Also, since

$\tan^{-1}\frac{y}{x}=\tan^{-1}\frac{-1}{1}=\tan^{-1}(-1)=\frac{-\pi}{4}$, our angle θ has to be related $\frac{-\pi}{4}$. However, considering the quadrant our point is in with respect to the xy-plane and given that we want our angle to be between 0 and 2π, we conclude that $\theta=\frac{7\pi}{4}$. Our calculation for φ is a little more straight forward. We have $\varphi=\cos^{-1}\left(\frac{z}{\rho}\right)=\cos^{-1}\left(\frac{-\sqrt{2}}{2}\right)=\frac{3\pi}{4}$.

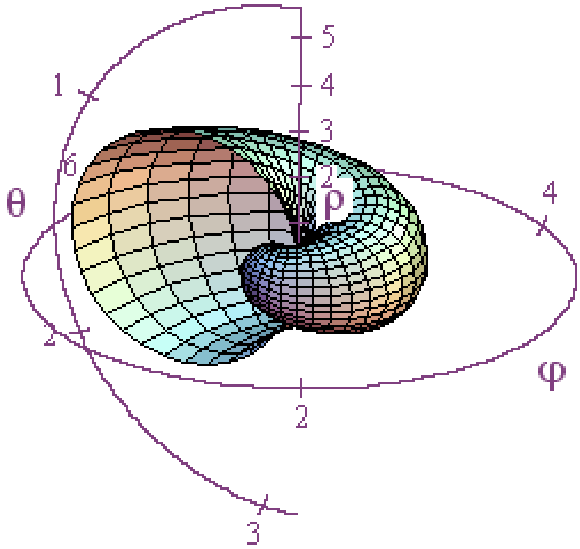
Hence, $(1,-1,-\sqrt{2})_{\text{rectangular}} \to \left(2,\frac{7\pi}{4},\frac{3\pi}{4}\right)_{\text{spherical}}$.

<u>Example 11:</u> Now that we've done a few conversions, let's look at some graphs. As we've mentioned, just as it's easiest to describe a cylinder in cylindrical coordinates,

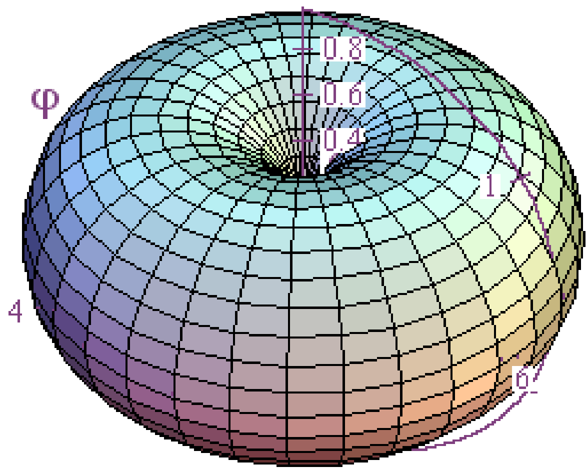
so are spherical coordinates the easiest way to describe a sphere. But of course, we can generate many other kinds of graphs, too, using spherical coordinates. However, they are often more difficult to analyze than graphs in good ol' rectangular *xyz*-coordinates. Nonetheless, let's think about $\rho = \sin\varphi \cdot \left(1.3^{\theta}\right)$. When $\theta = 0$, this becomes $\rho = \sin\varphi$, and in polar coordinates the graph of $r = \sin\theta$ is a circle of radius 1 that passes through the origin.



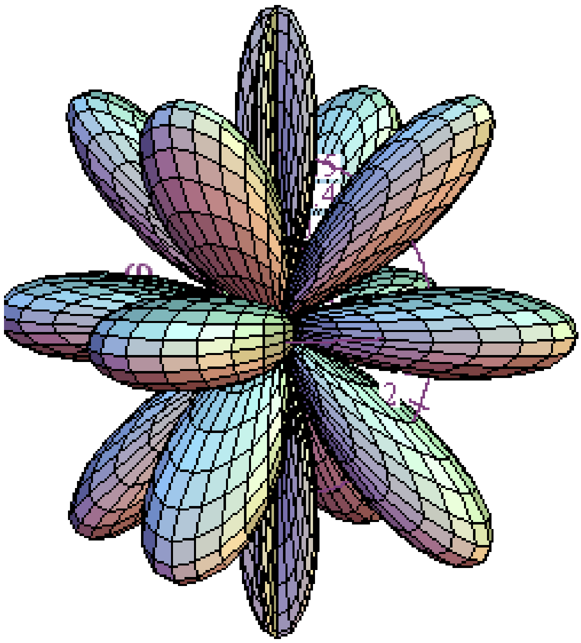
Thus, when $\theta = 0$ we expect the graph of $\rho = \sin\varphi \cdot \left(1.3^{\theta}\right)$ to be just a circle. However, as θ increases, ρ will also increase, and the radius of our circle gets larger. The end result is as follows.



If, on the other hand, we just graphed $\rho = \sin\varphi$ in three dimensions, then the radius of our circle would remain equal to one, and the following graph would result.



Example 12: For our next example, let's consider $\rho = \cos 4\varphi \cdot \cos 2\theta$. The first factor should cause oscillations in the graph as φ varies from 0 to π, and the second factor should create oscillations as we let θ go from 0 to 2π. The end result, as you can see below, is pretty ugly!

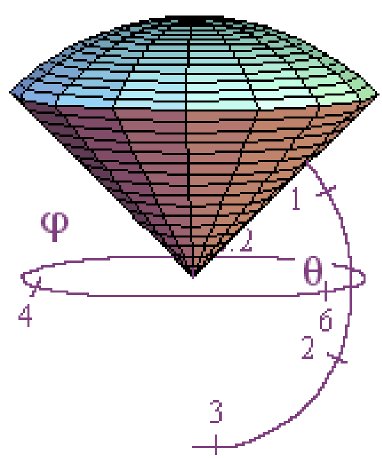


Example 13: And for our final example, let's see how we can create our filled ice cream cone in spherical coordinates. In this instance, it is particularly easy. The top part of our previous ice cream cone was a sphere of radius $\sqrt{2}$, and it extended downward $45°$ from the top. That means we want to let φ vary from 0 to $\pi/4$, and then let θ vary form 0 to 2π. And finally, for the cone part, we fix φ at $\pi/4$, let θ

vary form 0 to 2π , and let ρ vary from 0 to $\sqrt{2}$. A more succinct description of the two pieces is as follows.

Ice Cream Top	Cone
$\rho = \sqrt{2}$	$0 \le \rho \le \sqrt{2}$
$0 \le \theta \le 2\pi$	$0 \le \theta \le 2\pi$
$0 \le \varphi \le \frac{\pi}{4}$	$\varphi = \frac{\pi}{4}$

And the end result is the tasty graph below!



Finally, for future reference, here is a table that summarizes the relationships we talked about earlier between rectangular and cylindrical and spherical coordinates.

Cylindrical	Spherical
$x = r\cos\theta$	$x = r\cos(\theta) = \rho\sin(\varphi)\cos(\theta)$
$y = r\sin\theta$	$y = r\sin(\theta) = \rho\sin(\varphi)\sin(\theta)$
$z = z$	$z = \rho\cos(\varphi)$
$x^2 + y^2 = r^2$	$\rho^2 = x^2 + y^2 + z^2$
$\tan\theta = y/x$	$\tan\theta = \frac{y}{x}$
	$\varphi = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{\rho}$

Warning! Danger, Will Robinson, danger! Notation for spherical coordinates is not fixed! Some write (ρ,θ,φ), some write (ρ,φ,θ), some substitute r for ρ and switch θ and φ, and some do things that no one wants to know about! Always understand what convention your author is following before proceeding.

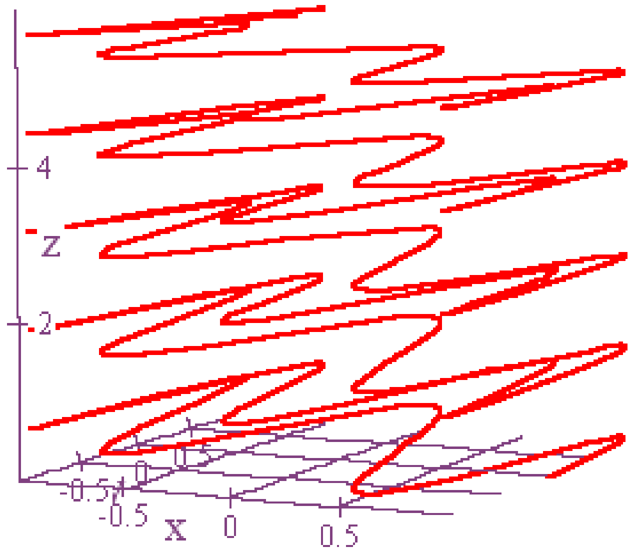
CHAPTER 3

PARAMETRIC EQUATIONS FOR CURVES IN SPACE

At this point we've talked an awful lot about graphs of functions of two variables and also about graphing surfaces in both cylindrical and spherical coordinates, and we've looked at an incredible number of examples of the kinds of surfaces that we can generate. However, sometimes it's not really a surface at all that we want to describe. Sometimes we just want to describe a curve or path that something like a baseball might travel when thrown. How do we do that? Well, the easiest way is usually to express the ball's location in space as a function of time. When we do this, we think of time, t, as a parameter for determining location of our object in coordinates (x, y, z). Consequently, we need our variables x, y, and z to all be expressed as functions of t. When we specify all three functions as well as the range of values for t, then we call the result our parametric equations for the curve that, in this case, our baseball will travel.

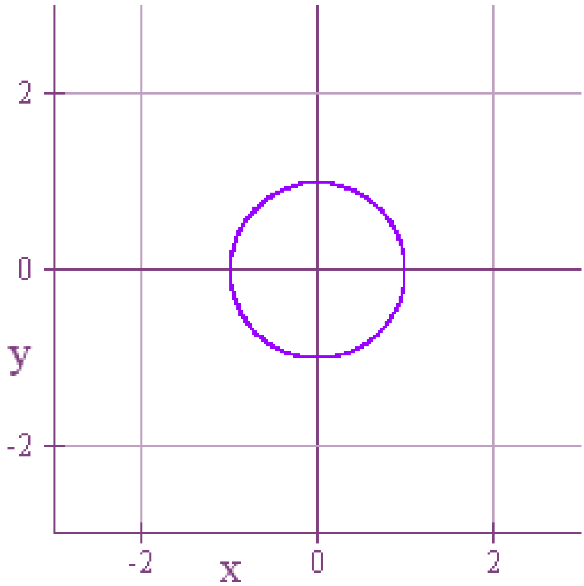
Now let's look at a simple example. Below are some parametric equations followed by the curve they produce.

$$
\begin{aligned}
&x = \cos(t) \\
&y = \sin(5t) \\
&z = \frac{t}{5} \\
&0 \le t \le 30
\end{aligned}
$$



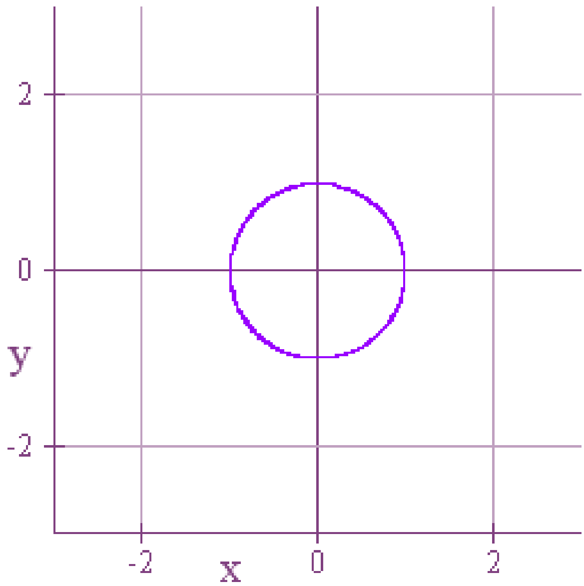
Well, that looks pretty disgusting! The good news is that we won't really need to know how to do very many parametrizations. In fact, there are only three things we'll need to know how to do well: (1) how to construct parametric equations for a line, (2) how to construct parametric equations for a circle, and (3) how to construct parametric equations for a cross-section of certain types of planes with a surface. Let's begin with the circle.

Fotunarely, you probably already know how to parametrize a circle – you just don't know that you know. Nevertheless, if you think back to trigonometry and polar coordinates, then you'll recall that for a circle of radius 1 with center at the origin, we have $x = \cos\theta$ and $y = \sin\theta$ where $0 \le \theta \le 2\pi$. We can also think of these equations as parametric equations where the parameter is θ. If we graph the corresponding curve in 2-dimensions, then we get our unit circle.



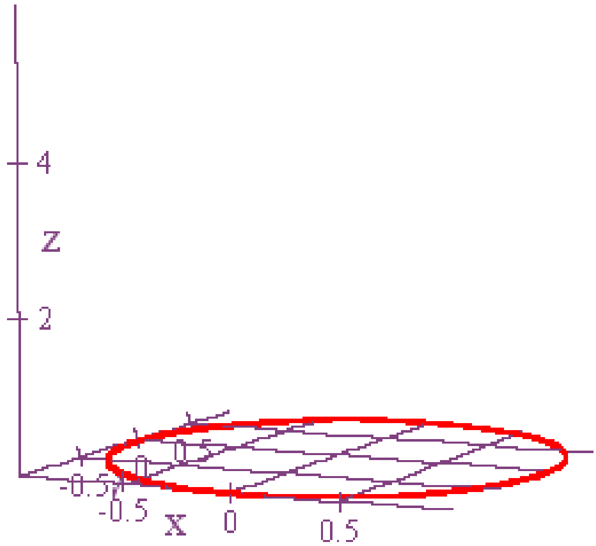
In this case, if we start with $\theta = 0$ and end with $\theta = 2\pi$, then we'll trace our circle in the counterclockwise direction both starting and ending at the point $(1,0)$. Much later on, we'll designate the counterclockwise direction as the positive direction, and we'll be very concerned about which direction our curve is traced in. For now, however, it won't be that much of a concern for us. One of the things we do want to take note of at this point, though, is that there will always exist an infinite number of parametrizations for any particular curve. For example, if in our parametric equations above, we replace θ by $\theta/2$ and change the range for θ to $0 \le \theta \le 4\pi$, then the end result is the same.

$$x = \cos(\theta/2)$$
$$y = \sin(\theta/2)$$
$$0 \le \theta \le 4\pi$$



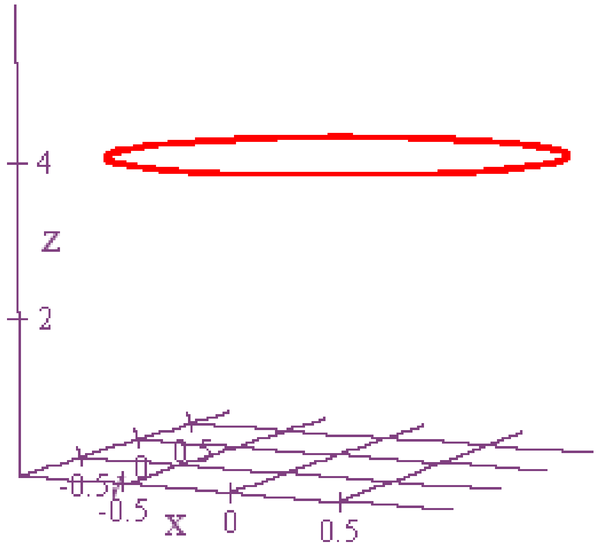
If we want to graph this circle in 3-dimensions but keep it in the *xy*-plane, then all we have do is to add the coordinate $z = 0$. By the way, at this point I'm going to switch to using *t* for the parameter.

$$x = \cos(t)$$
$$y = \sin(t)$$
$$z = 0$$
$$0 \le t \le 2\pi$$



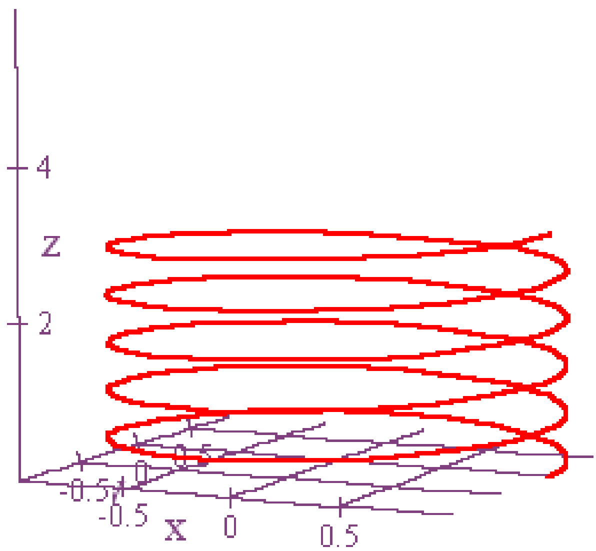
And if we want to elevate this circle to $z = 4$, all we have to do is change the fixed value of z to four.

$$x = \cos(t)$$
$$y = \sin(t)$$
$$z = 4$$
$$0 \le t \le 2\pi$$

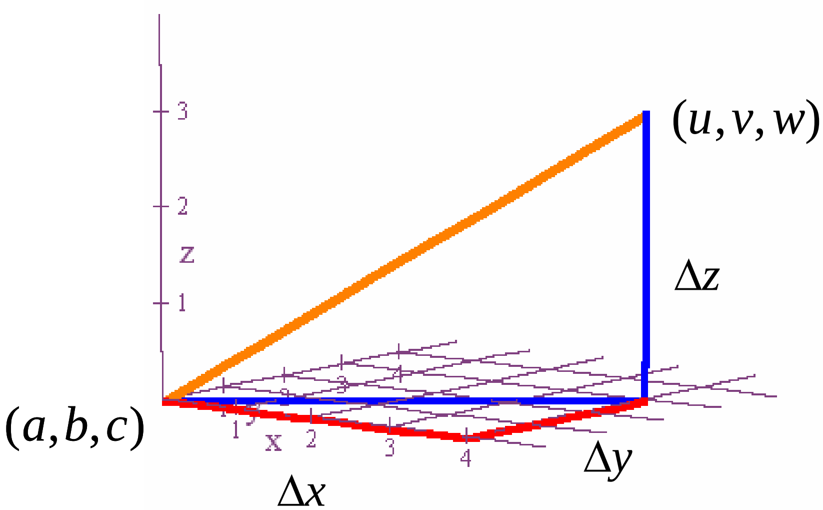


A fun variation we can do of a circle is a spiral that is technically known as a helix. However, I like to call it a slinky. To make a helix (or slinky), set your parametric equations so that x and y describe a circle, and then let z gradually increase as t increases.

$$x = \cos(t)$$
$$y = \sin(t)$$
$$z = t/10$$
$$0 \le t \le 10\pi$$



So that's it for circles! Now let's start looking at lines. Suppose you want to define a line segment parametrically that starts at (a,b,c) and ends at (u,v,w).



Then I claim that the parametric equations are as follows:

$$\begin{aligned} x &= a + t \cdot \Delta x \\ y &= b + t \cdot \Delta y \\ z &= c + t \cdot \Delta z \\ 0 &\le t \le 1 \end{aligned}$$

Certainly, when $t = 0$ we are at the point (a,b,c), and when $t = 1$ we add just the right amount of change to *x*, *y*, and *z* to take us to the point (u,v,w). Also, note the following two things. First, in our parametric equations we have *x*, *y*, and *z* all given as linear functions of *t*. Furthermore, by contemplating the diagram above, we should be able to convince ourselves that the object created by this parametrization will be a line. For example, when $t = \frac{1}{2}$, we will arrive at half our total change for *x*, *y*, and *z*, and the diagram suggests that we will have covered half the straight line distance between (a,b,c) and (u,v,w). Now let's do a particular example.

Example 1: Find parametric equations for the line segment from $(1,2,3)$ to $(4,7,5)$.

First, we write set *x*, *y*, and *z* equal to the coordinates of our starting point.

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

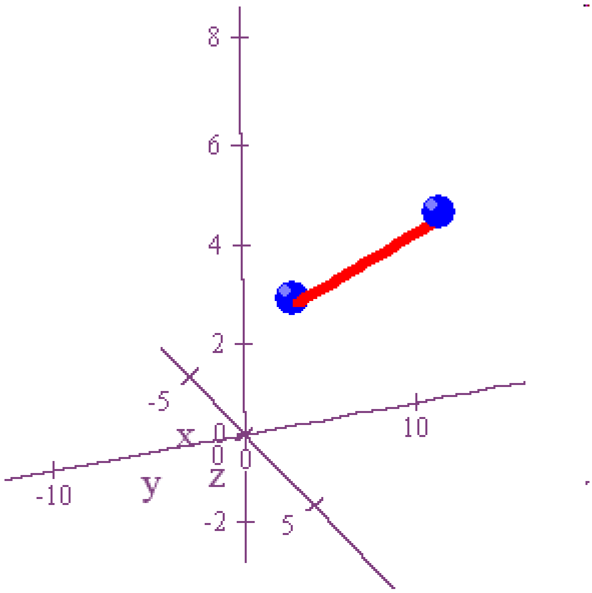
Next, we find the change associated with each variable.

$$\Delta x = 4 - 1 = 3$$
$$\Delta y = 7 - 2 = 5$$
$$\Delta z = 5 - 3 = 2$$

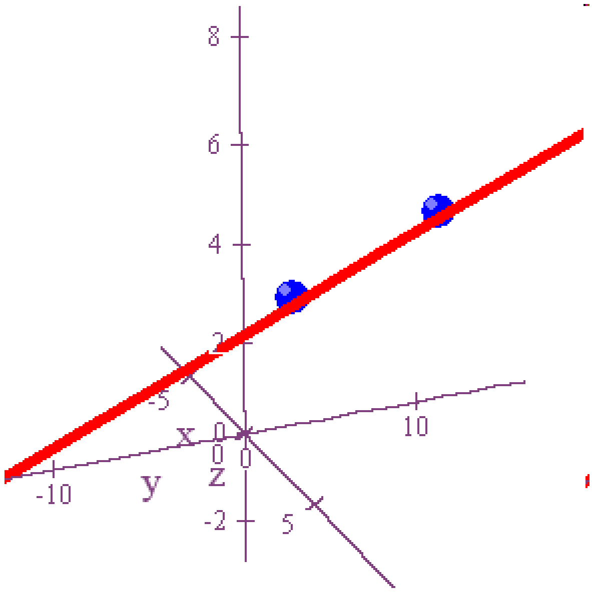
And finally, we multiply each change by *t*, add to our starting point, and let *t* vary from 0 to 1.

$$x = 1 + 3t$$
$$y = 2 + 5t$$
$$z = 3 + 2t$$
$$0 \le t \le 1$$

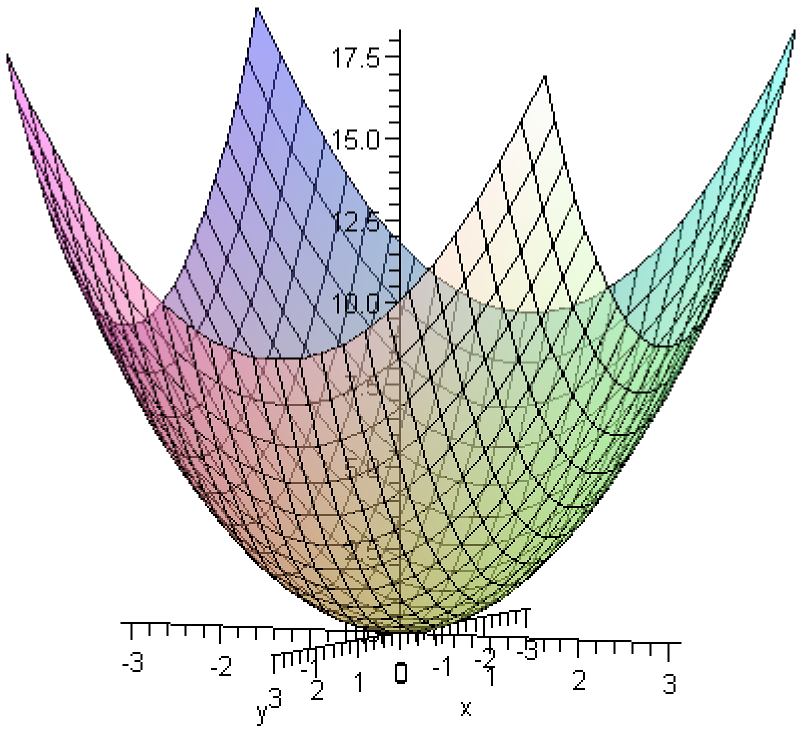
The end result is a very nice line segment from one point to another.



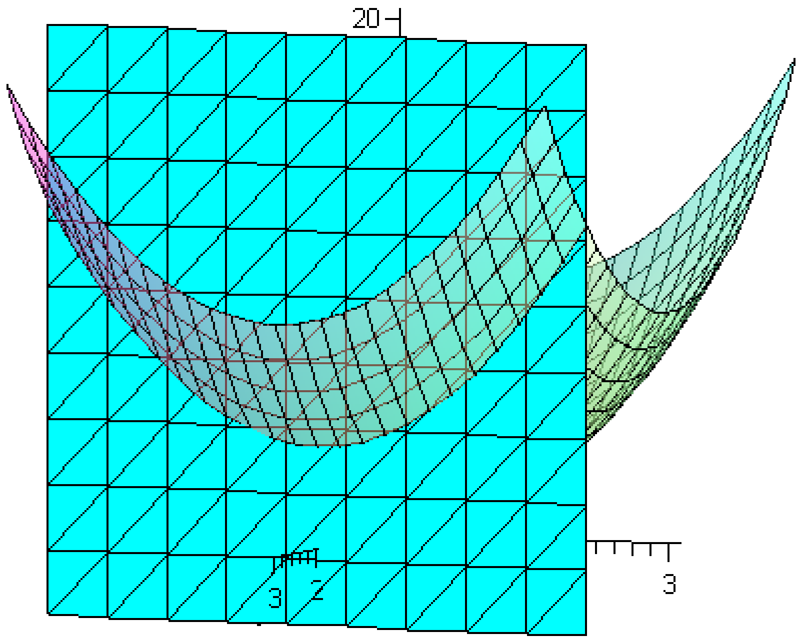
And if you want to extend the line, you just change the range for your parameter to $-\infty < t < \infty$.



And that's how you do a line! Now let's look at how to graph a curve of intersection along a surface. First, let's look at the graph of $z = x^2 + y^2$. That's our favorite paraboloid!

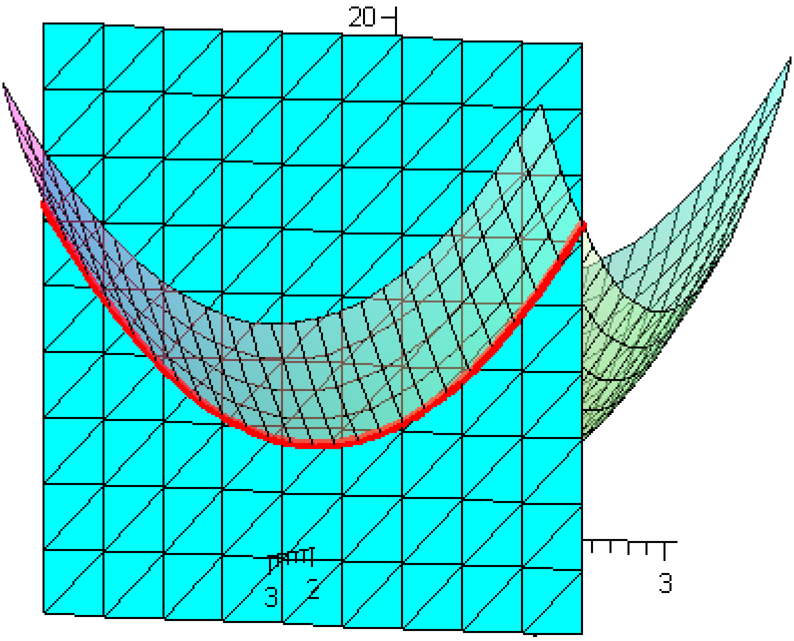


Now let's slice through this surface with the plane $x = 2$.

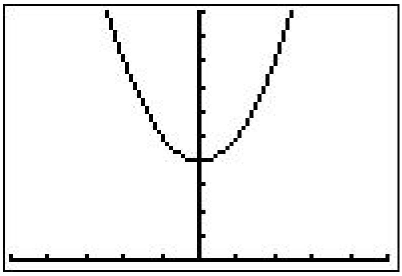


The result is a nice, parabolic cross-section. As we've seen before, we can get the equation for this cross-section by setting $x = 2$ in our formula for the parabolid. This gives us $z = 2^2 + y^2 = 4 + y^2$. To now create a parametrization for this curve in 3-dimensional space, one thing, at least, should be clear. We should fix *x* to the value 2. And now the rest of it is really quite simple. Jut set $y = t$ and then $z = 4 + t^2$. This is really just another way of saying that $z = 4 + y^2$. And finally, for the range of values of our parameter, for the graph above it will suffice to use $-3 \le t \le 3$ since that is the range I used in the graph for *x* and *y*. Okay, we're now ready to look at the parametric equations and the resulting graph.

$$x = 2$$
$$y = t$$
$$z = 4 + t^2$$
$$-3 \le t \le 3$$

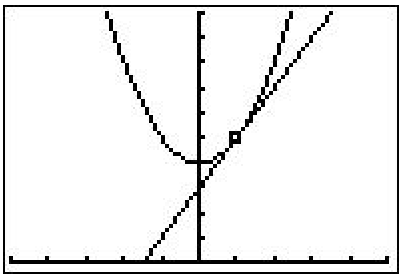


Success! However, we can do even more. Suppose we look at the graph of our cross-section $z = 4 + y^2$ in just 2-dimensions. Then we get something like the following.



If we set $y = 1$ on this graph, then $z = 4 + 1^2 = 5$, and we can use standard calculus techniques to find the slope of the tangent line to this curve at the point $(1,5)$.

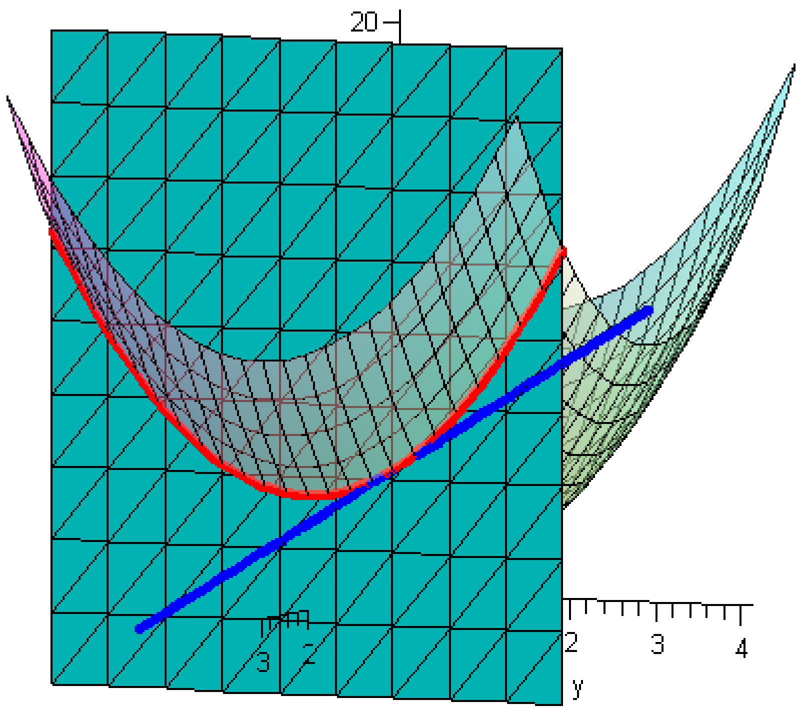
Clearly, $\frac{dz}{dy} = 2y$ and $\left.\frac{dz}{dy}\right|_{y=1} = 2$. Thus, an equation for our tangent to this 2-dimensional curve is $z = 2(y-1)+5 = 2y+3$.



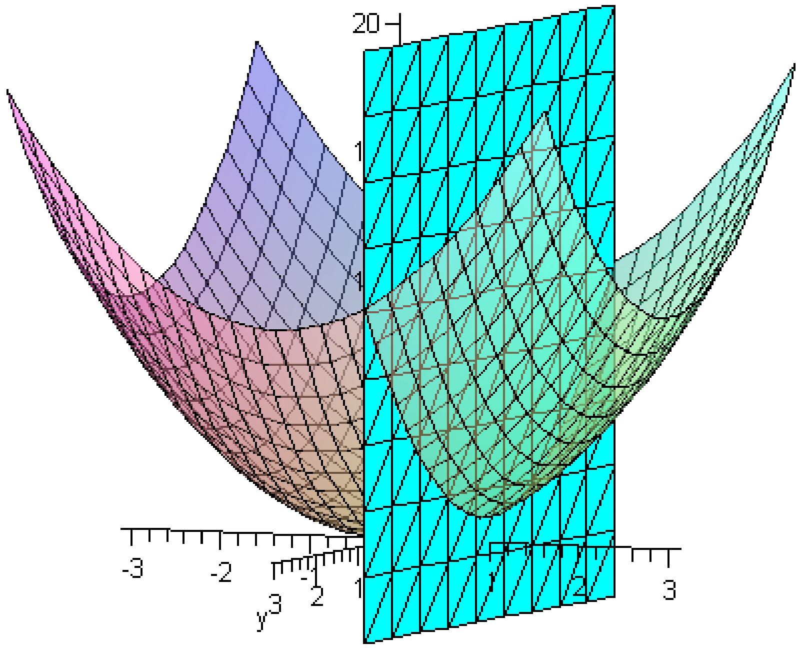
Now the question we want to ask ourselves is can we easily move all of this back onto our 3-dimensional surface, and the answer is yes, if we define our tangent line parametrically. First of all, since our point in 2-dimensions was $(y,z) = (1,5)$ and since we want everything to wind up in the plane $x = 2$, the point on the surface at which we want to place our tangent line is $(x,y,z) = (2,1,5)$. Second, since the slope of our tangent line in 2-dimensions was 2, in our parametric equations all we need to do is make sure that *z* grows twice as fast as *y*. All of this will happen perfectly if we set our parametric equations to:

$$x = 2$$
$$y = 1 + t$$
$$z = 5 + 2t$$
$$-\infty < t < \infty$$

Notice that these are parametric equations for a line that passes through $(2,1,5)$, the *x*-value is permanently fixed at 2 which will put our line in the plane $x = 2$, and *z* grows twice as fast as *y*. Now let's look at the result.

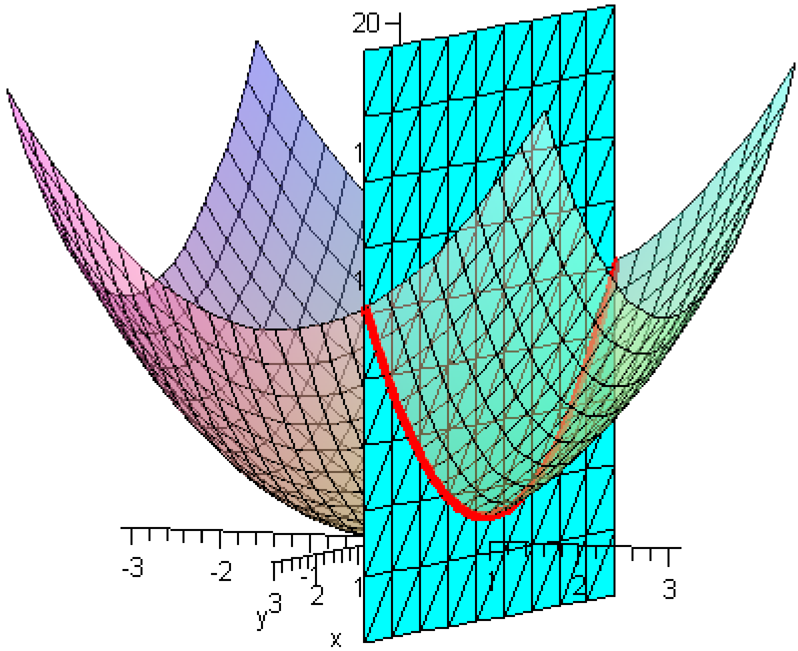


Perfect! Now let's do exactly the same thing, but this time we'll fix the *y*-value. Since the point at which we want to construct a tangent line to the surface is $(2,1,5)$, we'll intersect our surface with the plane $y = 1$.



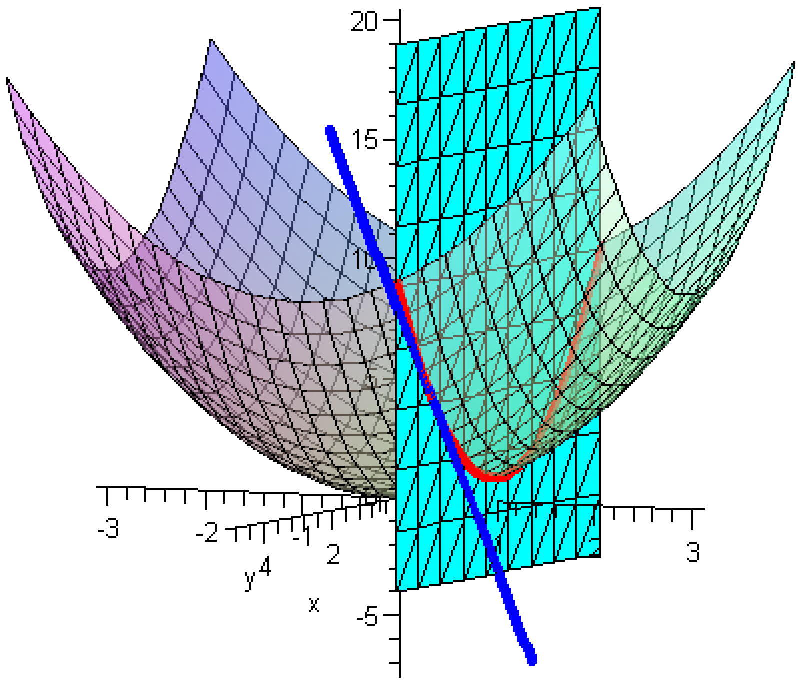
So far so good! Now let's add the graph of the cross-section to the surface. In this case, we'll have $z = x^2 + 1^2 = x^2 + 1$, and so our parametric equations should be:

$$\begin{aligned} x &= t \\ y &= 1 \\ z &= t^2 + 1 \\ -3 &< t < 3 \end{aligned}$$

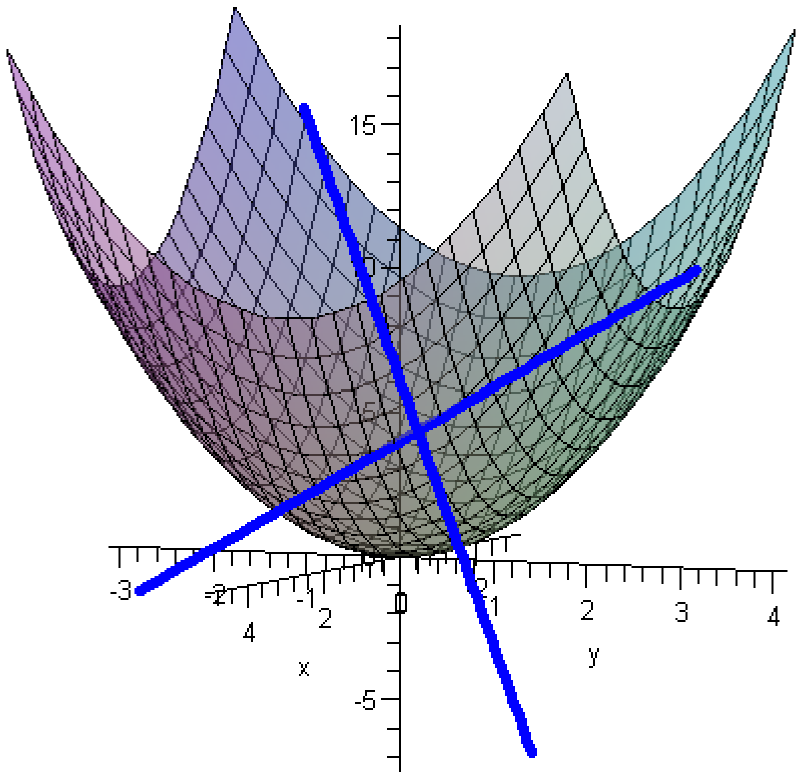


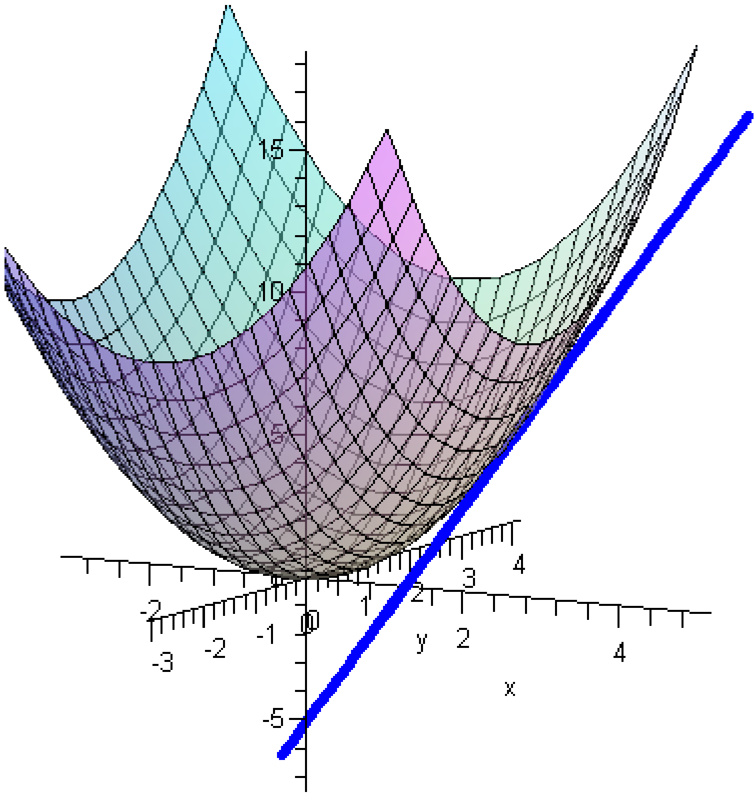
Most excellent! Now let's see if we can add the tangent line to the point $(2,1,5)$ that lies in the plane $y=1$. To get the slope of the tangent line, we'll consider $z=x^2+1$ as a function of one variable and differentiate to get $\frac{dz}{dx}=2x$. If we evaluate this derivative at $x=2$, we get that the slope of our tangent line is 4. We don't really need to construct the equation in 2-dimensions as we did previously since all we need is the slope in order to finish setting up parametric equations for the line in 3-dimensions. Just remember that, in this problem, if *x* increases by *t*, then *z* has to increase by 4*t* so that the slope of the line will be 4. And the parametric equations for this second tangent line at the point $(2,1,5)$ are:

$$\begin{aligned} &x = 2 + t \\ &y = 1 \\ &z = 5 + 4t \\ &-\infty < t < \infty \end{aligned}$$

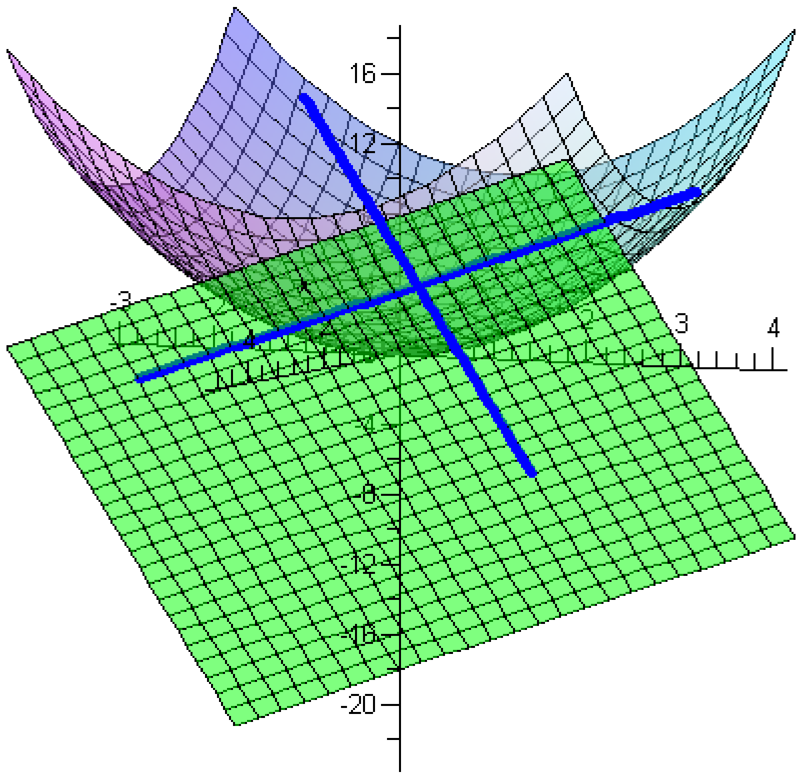


Ah, yes, perfection is a beautiful thing! If we now, however, look at the graph of our original parabolid with only the two tangent lines at the point $(2,1,5)$, then we might notice something very important. Namely, these two lines define a unique plane that is tangent to our surface at the point $(2,1,5)$. As we proceed through this book this notion of a tangent plane will become increasingly important as it is simply the higher dimensional version of the tangent line that you undoubtedly studied in first semester calculus.


15
-5
-3
-2
-1
0
1
2
3
4
x
y



The best question now that we can ask ourselves is whether or not we know enough to find an equation for this tangent plane. Fortunately, the answer is yes! What we do know is that slope of the tangent line in the direction of the positive *x*-axis is 4, the slope of the tangent line in the direction of the positive *y*-axis is 2, and the point $(2,1,5)$ is in the tangent plane. Furthermore, we know that $z = Ax + By + C$ is an equation for a plane and that the coefficients of *x* and *y* correspond to slopes of tangent lines in directions of positive *x* and positive *y*, respectively. Thus, plugging in what we know, we get $5 = 4(2) + 2(1) + C \Rightarrow C = -5$. Hence, the equation for the tangent plane is $z = 4x + 2y - 5$.


16
12
-3
3
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-4
-8
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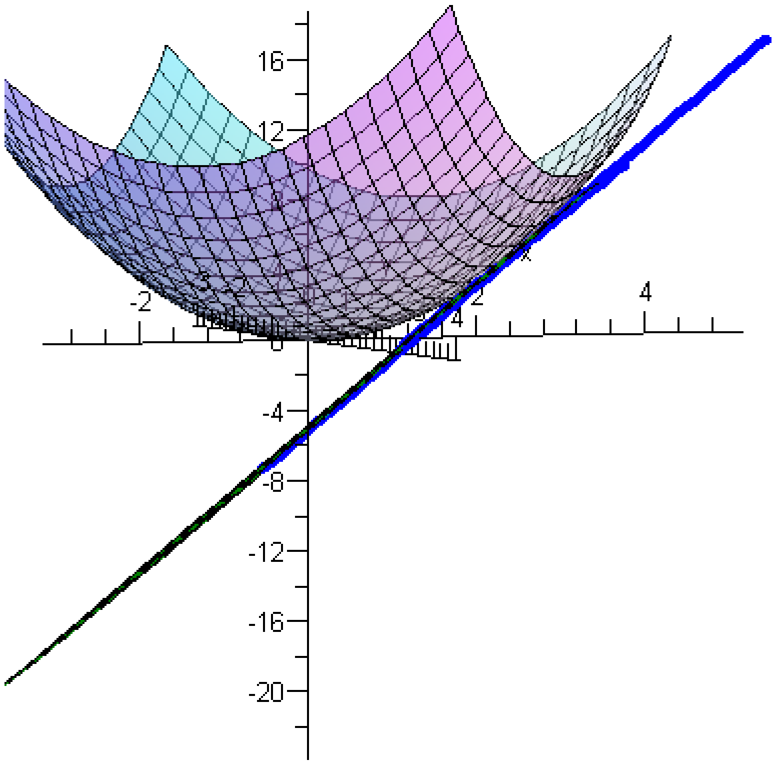
-16
-20

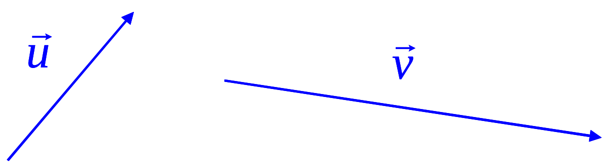


I love it when a construction comes together!

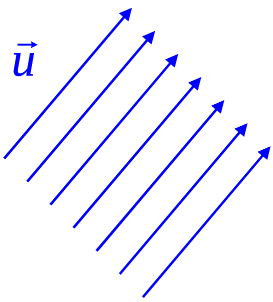
CHAPTER 4

VECTORS

Before we go any further, we must talk about vectors. They are such a useful tool for the things to come. The concept of a vector is deeply rooted in the understanding of physical mechanics and how physicists view forces in the universe. A force tends to act in a particular direction and with its own particular magnitude, and this is the basis for the notion of a vector. A vector is, thus, generally thought of as a quantity that possesses both direction and magnitude. In terms of notation, we often put a small arrow above a letter to indicate that we want it to represent a vector. Hence, we write $\vec{u}$ or $\vec{v}$ to indicate that we are speaking of vectors. Graphically, we draw arrows to represent vectors. In this geometric definition, the length of the arrow corresponds to the magnitude of the vector, and the direction the arrow points in represents the direction of the vector. Thus, we'll also work with graphical representations such as the ones below.

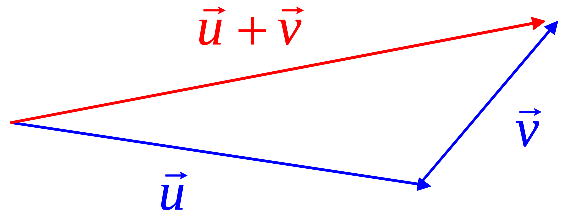


An interesting consequence of our definition of a vector as something possessing magnitude and direction is that vectors aren't defined by their location in space. In other words, two arrows that point in the same direction and have the same magnitude represent the same vector regardless of their location. This may seem strange not to tie a vector to a location, but it actually works out better this way in practice because to add vectors together geometrically, we are going to have to move them around. Think of it like this. Suppose vectors are Lego pieces and that we add two pieces by connecting them together. In that case, we wouldn't consider a Lego piece as losing its identity simply because we picked it up and moved it around. It's the same way with vectors. If, in a geometric sense, I pick one up and move it to a different location, then it's still the same vector as long as it has the same length and points in the same direction. Hence, each of the arrows below represents the same vector $\vec{u}$.

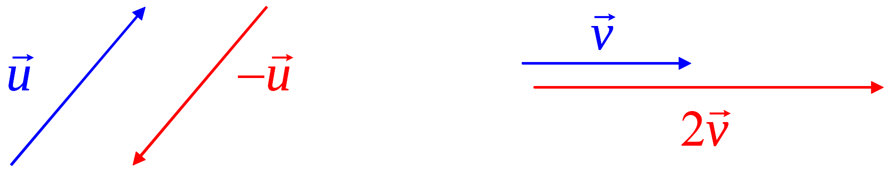


Again, though, you might object by saying that when something pushes against you, that force is not only acting in a particular direction with a particular magnitude, it's also being applied at a particular location in space. Well, that's absolutely true, and we have another concept to cover that case. When we do tie vectors to locations in space, we call the resulting configuration a vector field, and we'll deal extensively with vector fields much later on. But for now, we'll just think of vectors like the Lego pieces that we can move around and place wherever we want.

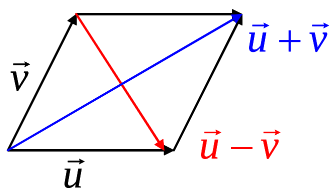
To add two vectors together geometrically, we simply place the starting point of one vector at the stopping point of the other vector, and then draw a single vector from the first starting point to the ultimate stopping point. The end result looks like this.



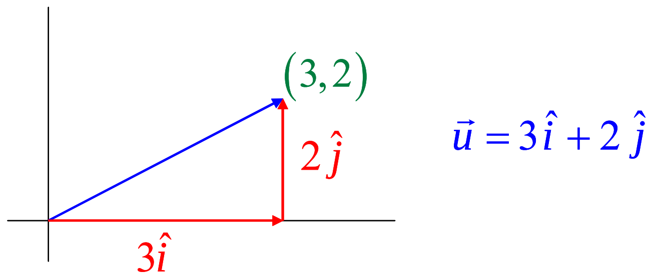
Also, given a vector $\vec{u}$, we define $-\vec{u}$ as the vector with the same magnitude, but pointing in the opposite direction. Additionally, we define something such as $2\vec{u}$ as the vector pointing in the same direction as $\vec{u}$, but with twice the length. When we multiply a vector times a number, we call that number a scalar, and in general, if one vector is a scalar multiple of another, then we say that the two vectors are parallel.



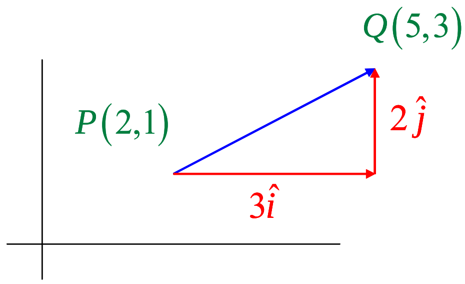
One final diagram worth looking at is the parallelogram formed by two vectors. In this drawing, one of the diagonals represents the sum of the two vectors, and the other diagonal represents the difference.



Well, drawing little diagrams made of arrows is loads of fun, but a very tedious way to do math. We really need a more algebraic way to denote and do operations with vectors, and to accomplish this task we'll define three very special unit vectors, i.e. vectors of length one unit. We'll call these vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$. The vector $\hat{i}$ has length 1 and points in the direction of the positive *x*-axis, the vector $\hat{j}$ has length 1 and points in the direction of the positive *y*-axis, and the vector $\hat{k}$ has length 1 and points in the direction of the positive *z*-axis. The diagram below now illustrates how, in 2-dimensions, we might represent a vector as the sum of its $\hat{i}$ and $\hat{j}$ components.



Notice, too, that if we place the starting point of a vector at the origin, then the coordinates of the stopping point correspond to the $\hat{i}$ and $\hat{j}$ components of the vector. Also, the more formal terms for starting and stopping points of a vector are either initial point and terminal point or tail and head of the vector. When we place a vector so that its initial point is at the origin, then we call that vector a <u>position vector</u>, and there is a natural association between the terminal point of a position vector and the vector's $\hat{i}$ and $\hat{j}$ components. If instead of starting our vector at the origin, we begin at a point *P* and terminate at a point *Q*, then we call that vector the displacement vector $\overrightarrow{PQ}$, and to get the $\hat{i}$ and $\hat{j}$ components of that vector we just do some subtraction with the corresponding *x* and *y* coordinates.



$$\overrightarrow{PQ} = (5-2)\hat{i} + (3-1)\,\hat{j} = 3\hat{i} + 2\,\hat{j}$$

Everything we've done above easily extends into three dimensions by merely adding in the $\hat{k}$ component for a true 3-dimensional representation. Furthermore, we can now easily do arithmetic with vectors just by performing the operations on their components. In other words, to add or subtract two vectors, just add or subtract their $\hat{i}$, $\hat{j}$, and $\hat{k}$ components, and to multiply a vector times a scalar, just multiply each component by that scalar. Below are a few simple examples.

Examples: Let $\vec{u} = 2\hat{i} + 3\,\hat{j} - 5\hat{k}$ and let $\vec{v} = -\hat{i} + 4\,\hat{j} + 2\vec{k}$. Then,

$$\vec{u} + \vec{v} = (2-1)\hat{i} + (3+4)\,\hat{j} + (-5+2)\hat{k} = \hat{i} + 7\,\hat{j} - 3\hat{k}$$

$$\vec{u} - \vec{v} = (2+1)\hat{i} + (3-4)\,\hat{j} + (-5-2)\hat{k} = 3\hat{i} - \hat{j} - 7\hat{k}$$

$$2\vec{u} = 2(2\hat{i} + 3\,\hat{j} - 5\hat{k}) = 4\hat{i} + 6\,\hat{j} - 10\hat{k}$$

Let's now talk about how we might find the length of a vector algebraically. Again, let $\vec{u} = 2\hat{i} + 3\hat{j} - 5\hat{k}$. If we think of this as a position vector that has its initial point at the origin, then its terminal point is $(2,3,-5)$, and thus, its length is just the distance that this point is from the origin. By the distance formula (which follows from the Pythagorean Theorem) we get that this is $\sqrt{2^2 + 3^2 + (-5)^2} = \sqrt{4+9+25} = \sqrt{38}$. We usually denote the length of a vector by putting either absolute value signs or double absolute value signs around the vector to form either $|\vec{u}|$ or $\|\vec{u}\|$. I'm going to adopt the latter notation because I think it looks so much cooler! Thus, if $\vec{u} = 2\hat{i} + 3\hat{j} - 5\hat{k}$, then $\|\vec{u}\| = \sqrt{38}$. Additionally, there are going to be many times that we might want to construct a unit vector pointing in the same direction as our original vector. To accomplish this, all we need to do is divide our original vector by its length, and the result will be a vector of length 1 pointing in the same direction.

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{1}{\|\vec{u}\|} \cdot \vec{u} = \frac{1}{\sqrt{38}}\left(2\hat{i} + 3\hat{j} - 5\hat{k}\right) = \frac{2}{\sqrt{38}}\hat{i} + \frac{3}{\sqrt{38}}\hat{j} - \frac{5}{\sqrt{38}}\hat{k}$$

We also like to call this unit vector *the direction of the vector* $\vec{u}$. Generally, the easiest way to talk about the direction of any particular vector is to specify the unit vector that points the same way.

One more thing I should probably mention before moving on is an alternate notation that one can use for vectors written in terms of their $\hat{i}$, $\hat{j}$, and $\hat{k}$ components.

Instead of writing it as above, we can merely enclose the components inside brackets to denote the vector as in $\left\langle \frac{2}{\sqrt{38}}, \frac{3}{\sqrt{38}}, -\frac{5}{\sqrt{38}} \right\rangle = \frac{2}{\sqrt{38}}\hat{i} + \frac{3}{\sqrt{38}}\hat{j} - \frac{5}{\sqrt{38}}\hat{k}$. I don't use this notation all that often, but it does sometimes come in handy. Additionally, some people like to enclose the components inside parentheses rather than brackets, but this can cause some minor confusion regarding whether we are talking about a vector or a point.

Well, we're on a roll now! Representing a vector in terms of components makes vector arithmetic a lot easier, and we don't need to try and draw arrows in three dimension in order to do it. Now let's look at something more advanced. Namely, how we might multiply vectors together.

This may be hard to believe, but every now and then you will come across something in mathematics that looks totally weird, and you'll wonder how people ever came up with something like that. Vector multiplication is going to be a good example of such weirdness. We actually have two ways of multiplying vectors, and they are known, respectively, as the dot product and the cross product. Both of these methods were chanced upon in 1773 by the great mathematician Joseph Lagrange. He was working on a problem involving tetrahedrons, and apparently needed to utilize what we call today the dot product and the cross product in order to arrive at his solution. For

Lagrange these two product were most likely simply a means to an end in the course of his problem solving, but as often happens in mathematics, one person discovers something, and then others find a plethora of applications to other situations. Consequently, mathematicians occasionally chance upon something that looks weird and unusual, but if we find a lot of uses for it, then we keep it! And that's certainly the case with the dot product and the cross product.

So let's begin with the dot product (denoted by $\vec{a} \bullet \vec{b}$ and read "$\vec{a}$ dot $\vec{b}$"). What exactly is the dot product? Well, we'll define it this way.

<u>Definition:</u> If $\vec{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$ and $\vec{b} = b_1\,\hat{i} + b_2\,\hat{j} + b_3\,\hat{k}$, then the dot product is $\vec{a} \bullet \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

What the #$@*^? Well, trust me. This is going to turn out to be very useful. But first, a couple of comments. Notice that the dot product of two vectors is going to be a scalar, a number, and not another vector. Thus, if $\vec{u} = 2\hat{i} + 3\hat{j} - 5\hat{k}$ and $\vec{v} = -\hat{i} + 4\hat{j} + 2\vec{k}$, them $\vec{u} \bullet \vec{v} = (2)(-1) + (3)(4) + (-5)(2) = -2 + 12 - 10 = 0$. I wasn't really expecting the dot product to equal zero in this example, but we'll find out soon why having a dot product equal to zero is especially cool. Another thing we should notice about the dot product is that it is commutative. In other words, $\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a}$. We can prove this just by appealing to the definition and using the fact that multiplication of ordinary numbers is commutative. Thus,

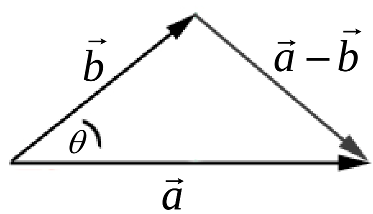
$\vec{a}\bullet\vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3 = \vec{b}\bullet\vec{a}$. Another interesting fact is that we also think of the dot product as an example of matrix multiplication. In other words, if $\vec{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$ and $\vec{b} = b_1\,\hat{i} + b_2\,\hat{j} + b_3\,\hat{k}$, then

$$\vec{a}\bullet\vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

Now we need to prove a theorem that will reveal some of the usefulness of the dot product.

<u>Theorem:</u> If $\vec{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$ and $\vec{b} = b_1\,\hat{i} + b_2\,\hat{j} + b_3\,\hat{k}$, then $\vec{a}\bullet\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$, where θ is the angle between the two vectors.

<u>Proof:</u> Let's start by looking at the triangle formed by the vectors $\vec{a}$ and $\vec{b}$. It might look something like the following.



Also, according to the Law of Cosines, we'll have $\|\vec{a}-\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta.$

But this implies that

$(a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2 = a_1{}^2+a_2{}^2+a_3{}^2+b_1{}^2+b_2{}^2+b_3{}^2-2\|\vec{a}\|\|\vec{b}\|\cos\theta.$

Expanding the left side of this equation yields

$a_1{}^2-2a_1b_1+b_1{}^2+a_2{}^2-2a_2b_2+b_2{}^2+a_3{}^2-2a_3b_3+b_3{}^2$

$=a_1{}^2+a_2{}^2+a_3{}^2+b_1{}^2+b_2{}^2+b_3{}^2-2\|\vec{a}\|\|\vec{b}\|\cos\theta.$ Next, subtracting like terms from each side gives us $-2a_1b_1-2a_2b_2-2a_3b_3=-2\|\vec{a}\|\|\vec{b}\|\cos\theta.$ And finally, dividing by -2 results in $a_1b_1+a_2b_2+a_3b_3=\|\vec{a}\|\|\vec{b}\|\cos\theta.$ Therefore, $\vec{a}\cdot\vec{b}=\|\vec{a}\|\|\vec{b}\|\cos\theta.$ ■

The immediate implication of this theorem is that we can use the dot product to find the angle between two vectors. In particular, $\dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}=\cos\theta \Rightarrow \theta=\cos^{-1}\left(\dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right).$ For example, if we go back to our previous vectors $\vec{u}=2\hat{i}+3\hat{j}-5\hat{k}$ and $\vec{v}=-\hat{i}+4\hat{j}+2\vec{k}$, then the angle between them is

$\theta=\cos^{-1}\left(\dfrac{\vec{u}\bullet\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)=\cos^{-1}\left(\dfrac{0}{\sqrt{38}\sqrt{21}}\right)=\cos^{-1}(0)=90°=\dfrac{\pi}{2}.$ In other words, the two vectors are actually perpendicular to one another, and this leads to an important corollary.

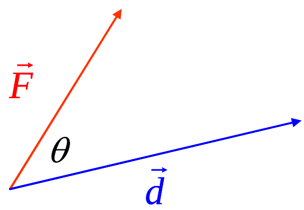
<u>Corollary:</u> Two vectors $\vec{a}$ and $\vec{b}$ are perpendicular if and only if $\vec{a} \bullet \vec{b} = 0$.

We now have an application of the dot product that is far from trivial. We can now easily find the angle between two vectors, and we can easily determine when two vectors are perpendicular (or orthogonal, as we also say). Another quick application of the dot product is that $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$. In other words, the dot product of a vector with itself is equal to the square of its length. The proof is easy and is left to you. Just write everything in component form and do the math!

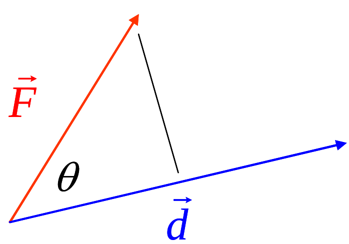
If the angle between two vectors is either 0 or π (equivalently, $0°$ or $180°$), then we say that the vectors are parallel. Another way to state this is to say that two vectors are parallel if and only if one vector is a scalar multiple of the other. For example, $\vec{u} = \hat{i} + \hat{j} + \hat{k}$, $\vec{v} = 2\vec{u} = 2\hat{i} + 2\hat{j} + 2\hat{k}$, and $\vec{w} = -\vec{u} = -\hat{i} - \hat{j} - \hat{k}$ are all parallel to one another.

Another application of the dot product that we'll use extensively later on is as a tool to compute work. If you've taken some physics, then you know that physicists have their own definition of work. The definition they generally use is $\text{work} = \text{force} \times \text{distance}$. This usually, but not always, corresponds well to our common sense notion of work. For example, if you have to use a lot of force to push something a great distance, that's a lot of work! But suppose for a moment that the direction in which the force is applied is not the same as the direction in which you are moving the object. What do you do then? That's where vectors help us!

Suppose we have two vectors $\vec{F} = \hat{i} + 2\hat{j}$ and $\vec{d} = 4\hat{i} + \hat{j}$.



The vector $\vec{d} = 4\hat{i} + \hat{j}$, in his case, represents the path that a force $\vec{F} = \hat{i} + 2\hat{j}$ is going to move something along, and we want to compute the work done. If you wish, think of the force as being measured in pounds and the distance as being measured in feet. The work then, as a product of the two, will have units of foot-pounds. What we need to do now is to figure out the component of our force vector that is parallel to the vector $\vec{d} = 4\hat{i} + \hat{j}$.



Using some trigonometry, that component is

$comp_{\vec{d}}\vec{F} = \|\vec{F}\|\cos\theta = \dfrac{\|\vec{F}\|\|\vec{d}\|\cos\theta}{\|\vec{d}\|} = \dfrac{\vec{F}\bullet\vec{d}}{\|\vec{d}\|} = \dfrac{6}{\sqrt{17}} = \dfrac{6\sqrt{17}}{17}$. The work done is now simply the component of $\vec{F}$ along the vector $\vec{d}$ times the length of the vector $\vec{d}$. In other words, $work = \left(comp_{\vec{d}}\vec{F}\right)\|\vec{d}\| = \left(\dfrac{\vec{F}\bullet\vec{d}}{\|\vec{d}\|}\right)\|\vec{d}\| = \vec{F}\bullet\vec{d} = 6\,\text{foot-pounds}$. The bottom line is that if a constant force $\vec{F}$ is moving an object in a straight line represented by a vector $\vec{d}$, then the work done is $\vec{F}\bullet\vec{d}$. Remember this one! It's important. Also, if we take the component of $\vec{F}$ along $\vec{d}$ and multiply it by a unit vector in the direction of $\vec{d}$, then we call that the projection of $\vec{F}$ onto $\vec{d}$. In particular, for the vectors $\vec{F} = \hat{i} + 2\hat{j}$ and $\vec{d} = 4\hat{i} + \hat{j}$ we have

$$proj_{\vec{d}}\vec{F} = \left(\|\vec{F}\|\cos\theta\right)\frac{\vec{d}}{\|\vec{d}\|} = \frac{\vec{F}\bullet\vec{d}}{\|\vec{d}\|}\cdot\frac{\vec{d}}{\|\vec{d}\|} = \left(\frac{\vec{F}\bullet\vec{d}}{\|\vec{d}\|^2}\right)\vec{d} = \left(\frac{\vec{F}\bullet\vec{d}}{\vec{d}\bullet\vec{d}}\right)\vec{d} = \frac{24}{17}\hat{i} + \frac{6}{17}\hat{j}.$$

Now let's look at how we define the cross product, $\vec{a}\times\vec{b}$, and trust me, this is really going to look crazy! Also, we read $\vec{a}\times\vec{b}$ as "$\vec{a}$ *cross* $\vec{b}$."

Definition: If $\vec{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$ and $\vec{b} = b_1\,\hat{i} + b_2\,\hat{j} + b_3\,\hat{k}$, then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{k}$$

$$= (a_2b_3 - a_3b_2)\,\hat{i} - (a_1b_3 - a_3b_1)\,\hat{j} + (a_1b_2 - a_2b_1)\,\hat{k}.$$

Cool! Three things we should notice right now,

1. The cross product is defined as the determinant of a matrix.
2. The cross product of two vectors results in another vector.
3. We have no idea at this point why this is going to be useful.

Let's start alleviating the problem highlighted in the last item above by proving a few theorems.

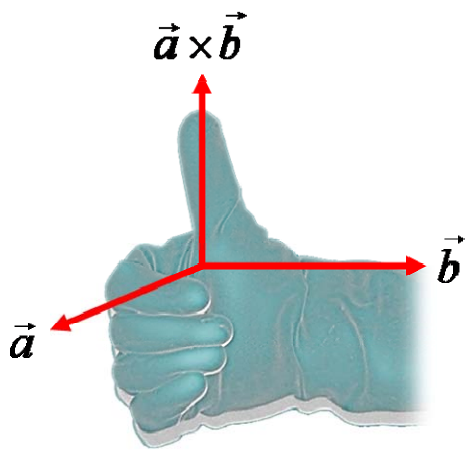
Theorem: The cross product of two vectors $\vec{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$ and $\vec{b} = b_1\,\hat{i} + b_2\,\hat{j} + b_3\,\hat{k}$ is perpendicular to both vectors.

Proof: Be definition, $\vec{a}\times\vec{b} = (a_2b_3 - a_3b_2)\,\hat{i} - (a_1b_3 - a_3b_1)\,\hat{j} + (a_1b_2 - a_2b_1)\,\hat{k}$. To show that $\vec{a}$ is perpendicular to this vector, we just need to show that their dot product is equal to zero. Hence,

$$\vec{a}\bullet\left(\vec{a}\times\vec{b}\right)=(a_1\,\hat{i}+a_2\,\hat{j}+a_3\,\hat{k})\bullet((a_2b_3-a_3b_2)\,\hat{i}-(a_1b_3-a_3b_1)\,\hat{j}+(a_1b_2-a_2b_1)\,\hat{k})$$

$=a_1a_2b_3-a_1a_3b_2-a_2a_1b_3+a_2a_3b_1+a_3a_1b_2-a_3a_2b_1=0$. A similar calculation shows that $\vec{b}\bullet\left(\vec{a}\times\vec{b}\right)=0$, and so it follows that the cross product $\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. ■

If our vectors $\vec{a}$ and $\vec{b}$ are not parallel, then they are going to define a plane, and the cross product $\vec{a}\times\vec{b}$ will be perpendicular to this plane. However, we still want more information because, for example, a vector perpendicular to a horizontal plane could point either up or down. Can we possibly determine in an easy way which direction our cross product will point in? Fortunately, the answer is yes. We just use what we call the right-hand rule.



If we point the fingers of our right hand in the direction of vector $\vec{a}$ and curl our fingers towards vector $\vec{b}$, then our thumb will point in the direction of $\vec{a}\times\vec{b}$. Notice, in particular, that if we apply this right-hand rule to the cross product $\vec{b}\times\vec{a}$, then the resulting vector points in the opposite direction from $\vec{a}\times\vec{b}$. This is very important because it tells us that, unlike the dot product, the cross product is not commutative. In other words, $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$. Thus, the order in which you do these things makes a difference. Also, the correct relationship between these cross products is $\vec{a}\times\vec{b}=-\left(\vec{b}\times\vec{a}\right)$.

Now let's look at some more theorems.

Theorem: $\left\|\vec{a}\times\vec{b}\right\|=\left\|\vec{a}\right\|\left\|\vec{b}\right\|\sin\theta,$ *where* θ *is the angle between the two vectors.*

Proof: The proof is amazingly simple!

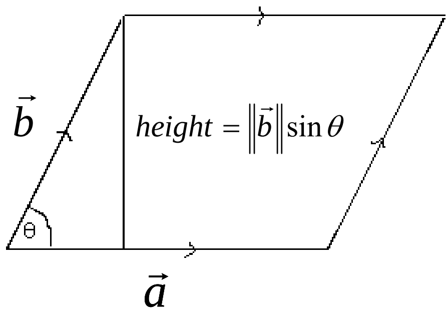
$$\left\|\vec{a}\times\vec{b}\right\|^2 = (a_2b_3 - a_3.b_2)^2 + (a_1b_3 - a_3.b_1)^2 + (a_1b_2 - a_2.b_1)^2$$
$$= a_2{}^2b_3{}^2 - 2a_2a_3b_2b_3 + a_3{}^2b_2{}^2 + a_1{}^2b_3{}^2 - 2a_1a_3b_1b_3 + a_3{}^2b_1{}^2$$
$$+ a_1{}^2b_2{}^2 - 2a_1a_2b_1b_2 + a_2{}^2b_1{}^2$$
$$= a_1{}^2b_2{}^2 + a_1{}^2b_3{}^2 + a_2{}^2b_1{}^2 + a_2{}^2b_3{}^2 + a_3{}^2b_1{}^2 + a_3{}^2b_2{}^2$$
$$- 2a_1a_2b_1b_2 - 2a_1a_3b_1b_3 - 2a_2a_3b_2b_3$$
$$= a_1{}^2b_1{}^2 + a_2{}^2b_2{}^2 + a_3{}^2b_3{}^2 + a_1{}^2b_2{}^2 + a_1{}^2b_3{}^2 + a_2{}^2b_1{}^2 + a_2{}^2b_3{}^2 + a_3{}^2b_1{}^2 + a_3{}^2b_2{}^2$$
$$- 2a_1a_2b_1b_2 - 2a_1a_3b_1b_3 - 2a_2a_3b_2b_3 - a_1{}^2b_1{}^2 - a_2{}^2b_2{}^2 - a_3{}^2b_3{}^2$$
$$= (a_1{}^2 + a_2{}^2 + a_3{}^2)(b_1{}^2 + b_2{}^2 + b_3{}^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$
$$= \|\vec{a}\|^2\left\|\vec{b}\right\|^2 - \left(\vec{a}\cdot\vec{b}\right)^2 = \|\vec{a}\|^2\left\|\vec{b}\right\|^2 - \left(\|\vec{a}\|\left\|\vec{b}\right\|\cos\theta\right)^2$$
$$= \|\vec{a}\|^2\left\|\vec{b}\right\|^2 - \|\vec{a}\|^2\left\|\vec{b}\right\|^2\cos^2\theta = \|\vec{a}\|^2\left\|\vec{b}\right\|^2\left(1-\cos^2\theta\right) = \|\vec{a}\|^2\left\|\vec{b}\right\|^2\sin^2\theta.$$

Thus, $\left\|\vec{a}\times\vec{b}\right\|^2 = \|\vec{a}\|^2\left\|\vec{b}\right\|^2\sin^2\theta,$ and since $\sin\theta \ge 0$ *for* $0 \le \theta \le \pi$, we have that

$\left\|\vec{a}\times\vec{b}\right\| = \|\vec{a}\|\left\|\vec{b}\right\|\sin\theta.$ Any questions??? ■

One application of this result is a formula in terms of vectors for the area of a parallelogram. The picture below should be pretty self-explanatory.

$$Area = \|\vec{a}\|\left\|\vec{b}\right\|\sin\theta = \left\|\vec{a}\times\vec{b}\right\|$$

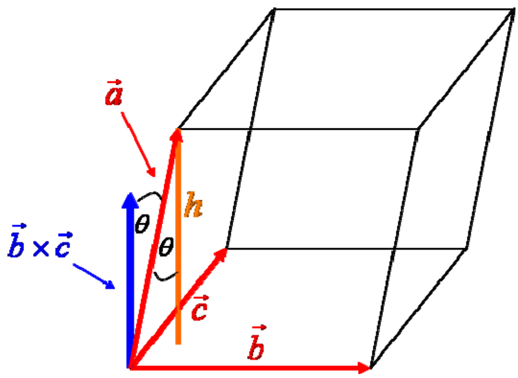


Let's look at an example of this. If $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{b} = -\hat{i} + 2\hat{j} - 2\hat{k}$, then the cross product of these two vectors is $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ -1 & 2 & -2 \end{vmatrix} = -14\hat{i} + 0\hat{j} + 7\hat{k} = -14\hat{i} + 7\hat{k}$.

Hence, the area of the parallelogram with sides $\vec{a}$ and $\vec{b}$ is

$Area = \left\|\vec{a} \times \vec{b}\right\| = \left\|-14\hat{i} + 7\hat{k}\right\| = \sqrt{245} = 7\sqrt{5} \approx 15.65$.

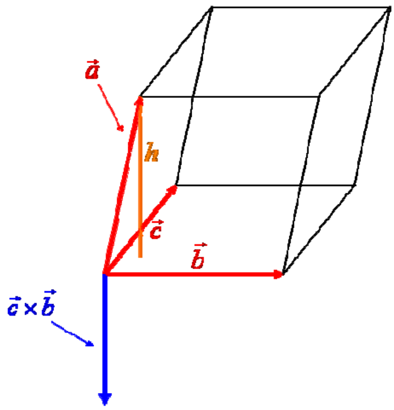
Now for another application. Suppose we have a parallelepiped whose base is a parallelogram defined by vectors $\vec{b}$ and $\vec{c}$, and the third defining side is vector $\vec{a}$.



Then $Area\,of\,base = \left\|\vec{b} \times \vec{c}\right\|$ and $height = \left\|\vec{a}\right\|\cos\theta$, where θ is the acute angle between $\vec{b} \times \vec{c}$ and $\vec{a}$. Then this suggests the following formula for the volume of the parallelepiped.

$$Volume = Area\,of\,base \times height = \left\|\vec{b} \times \vec{c}\right\| \cdot \left\|\vec{a}\right\|\cos\theta = \left(\vec{b} \times \vec{c}\right) \bullet \vec{a}$$

However, notice that if we don't take our cross product in just the right order, then θ will be an obtuse angle and $\cos\theta$ will be negative.

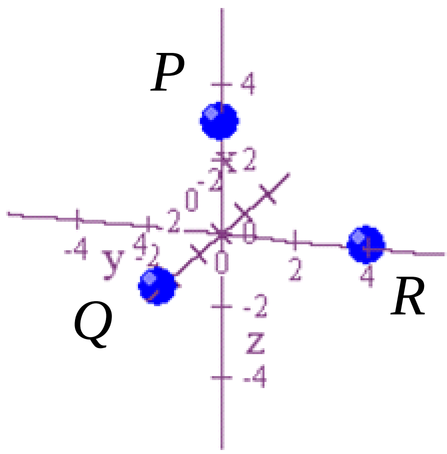


The easy fix for this is to not worry about the order and simply take the absolute value of our final result in order to ensure that we get back a positive number for the volume. Hence, $Volume = \left|\left(\vec{c}\times\vec{b}\right)\bullet\vec{a}\right| = \left|\left(\vec{b}\times\vec{c}\right)\bullet\vec{a}\right| = \left|\vec{a}\bullet\left(\vec{b}\times\vec{c}\right)\right|$. What may not be immediately apparent at this point, but is nonetheless easy for you to deduce is that $\left|\vec{a}\bullet\left(\vec{b}\times\vec{c}\right)\right| = absolute\,value\,of \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$. Hence, the volume of the parallelepiped defined by vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is the absolute value of $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$. Just do the math, and you'll see that this is correct.

Now let's look at a quick example. Suppose our parallelepiped is defined by the vectors $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} - 2\hat{k}$, and $\vec{c} = -\hat{i} - 3\hat{j} + 4\hat{k}$. Then the value of the determinant is $\begin{vmatrix} 2 & 3 & 4 \\ -1 & 2 & -2 \\ -1 & -3 & 4 \end{vmatrix} = 42$. Hence, the volume of the parallelepiped is

$|42| = 42$. What could be simpler!

Now let's look at a really important application. Remember that we have a special interest in being able to construct planes because in three dimensions the tangent plane to a point on a surface is what corresponds to a tangent line in two dimensions, and we already know how important tangent lines are to calculus. Thus, let's suppose that we have three distinct points that define a plane such as $P = (0,0,3)$, $Q = (5,0,0)$, and $R = (0,4,0)$.



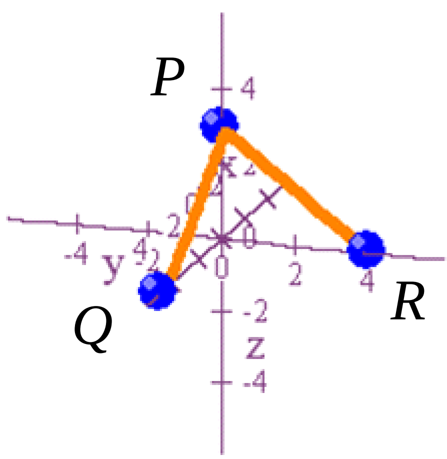
Then $\overrightarrow{PQ} = \langle 5, 0, -3 \rangle$ and $\overrightarrow{PR} = \langle 0, 4, -3 \rangle$ (notice that I've switched to an alternate notation in order to give you some practice with it). Also, we can construct parametric equations for line segments in order to add representations for these vectors to our graph. We'll parametrize the vector from P to Q as:

$$\begin{aligned} x &= 0 + 5t = 5t \\ y &= 0 + 0t = 0 \\ z &= 3 - 3t \\ 0 &\le t \le 1 \end{aligned}$$

And we can parametrize the vector from P to R as:

$$\begin{aligned} x &= 0 + 0t = 0 \\ y &= 0 + 4t = 4t \\ z &= 3 - 3t \\ 0 &\le t \le 1 \end{aligned}$$

And here is the picture we get,



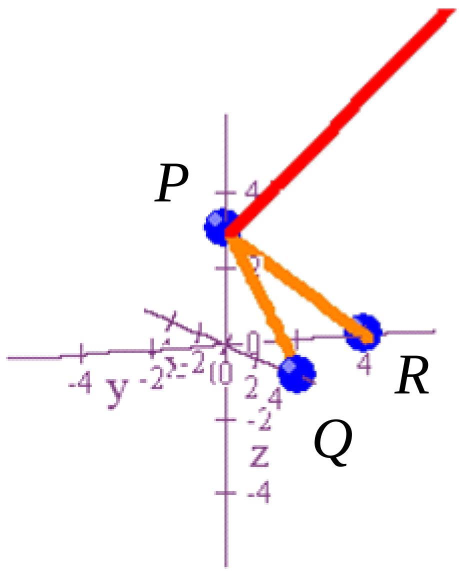
If we now take the cross product of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, then we will have a vector perpendicular to the plane defined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 5 & 0 & -3\\ 0 & 4 & -3\end{vmatrix}=12\hat{i}+15\hat{j}+20\hat{k}$$

Parametric equations for this cross product vector starting at point *P* are:

$$\begin{aligned}x&=0+12t=12t\\ y&=0+15t=15t\\ z&=3+20t\\ 0&\le t\le 1\end{aligned}$$

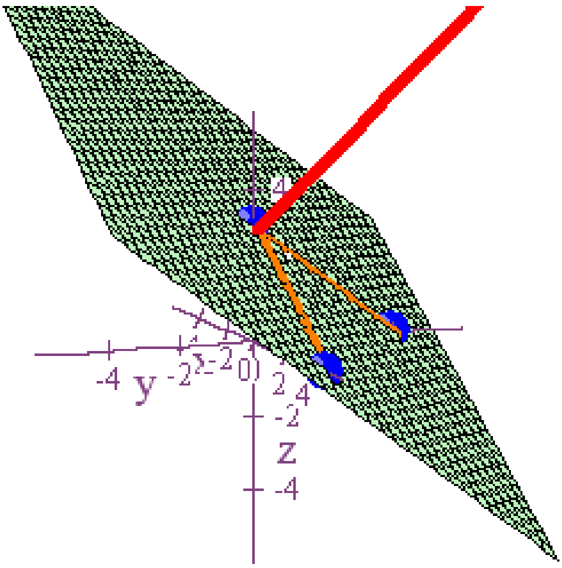
And here's what we get when we add this vector to the graph. We can certainly see that it looks perpendicular to the other two!



Now suppose that we have some point $X=(x,y,z)$ that lies in the plane defined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Then the vector $\overrightarrow{PX}=(x-0)\hat{i}+(y-0)\hat{j}+(z-3)\hat{k}$ lies in the plane, and it thus, must be perpendicular to the cross product $\overrightarrow{PQ}\times\overrightarrow{PR}=12\hat{i}+15\hat{j}+20\hat{k}$. Hence,

$\left(\overrightarrow{PQ}\times\overrightarrow{PR}\right)\bullet\overrightarrow{PX}=0\Rightarrow\langle 12,15,20\rangle\bullet\langle x,y,z-3\rangle=0\Rightarrow 12x+15y+20(z-3)=0$

$\Rightarrow 12x+15y+20z-60=0$. But this last equation is an equation for a plane, and if we solve it for z, then we can rewrite it as $z=-\frac{12}{20}x-\frac{15}{20}y+\frac{60}{20}=-\frac{3}{5}x-\frac{3}{4}y+3$. Let's graph this last equation and add it to what we have so far.



Wonderful! We can easily see that we've taken our three original points and successfully found an equation for the plane they define as well as a vector which is perpendicular to this plane. Speaking of the latter, notice that the components of our vector $\overrightarrow{PQ} \times \overrightarrow{PR} = 12\hat{i} + 15\hat{j} + 20\hat{k}$ appear also as coefficients of the equation for the plane, $12x + 15y + 20z - 60 = 0$. This is no accident. In fact, we can show that if we have an equation for a plane in the form $Ax + By + Cz + D = 0$, then the vector $A\hat{i} + B\hat{j} + C\hat{k}$ is perpendicular to this plane. How do we do that? I'll show you!

Suppose $Ax + By + Cz + D = 0$ is an equation for a plane. Then clearly $Ax + By + Cz = -D.$ Now suppose that (a,b,c) is a point in the plane. Then it also follows that $A \cdot a + B \cdot b + C \cdot c = -D$. Additionally, if (x, y, z) is another point in our plane, then the displacement vector from (a,b,c) to (x, y, z) is $\langle x-a, y-b, c-d \rangle.$ Let's now compute the dot product between this last vector and the vector $\langle A, B, C \rangle.$ In particular, we get $\langle A, B, C \rangle \bullet \langle x-a, y-b, z-c \rangle = A(x-a) + B(y-b) + C(z-c)$

$= Ax - A \cdot a + By - B \cdot b + Cz - C \cdot c = (Ax + By + Cz) - (A \cdot a + B \cdot b + C \cdot c) = -D + D = 0.$

Therefore, the two vectors are perpendicular, and since the point (x, y, z) was chosen arbitrarily from the plane, it follows that $\langle A, B, C \rangle$ is perpendicular to any vector in that plane, and hence, $\langle A, B, C \rangle$ is perpendicular to the plane $Ax + By + Cz + D = 0$.

If we summarize a few things we know at this point, then we can say that if $Ax + By + Cz + D = 0$ is any equation for a plane, then the coefficients *A*, *B*, and *C* define a vector perpendicular to that plane. Recall that the word orthogonal means the same as perpendicular. We also express this same idea by saying that the vector $\langle A, B, C \rangle$ is normal to the plane. Additionally, if our equation for the plane is written in the form $z = Ax + By + C$, then the coefficient *A* is the slope of the plane in the direction of the positive *x*-axis, and the coefficient *B* is the slope of the plane in the direction of the positive *y*-axis. Cool Stuff! And this will all be very important to us as we continue.

To wrap things up, we're now just going to give you a list of algebraic properties of vectors for future reference. We won't provide proofs of these properties, but they are all pretty straight forward if you just apply the definitions. Also, proving them is a fun thing to do at parties to amaze your friends and be a popular person!

If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and *c* and *d* are scalars, then,

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. $\vec{a} + \left(\vec{b} + \vec{c}\right) = \left(\vec{a} + \vec{b}\right) + \vec{c}$
3. $\vec{a} + \vec{0} = \vec{a}$
4. $\vec{a} + \left(-\vec{a}\right) = \vec{0}$
5. $c\left(\vec{a} + \vec{b}\right) = c\vec{a} + c\vec{b}$
6. $\left(c + d\right)\vec{a} = c\vec{a} + d\vec{a}$
7. $\left(cd\right)\vec{a} = c\left(d\vec{a}\right)$
8. $1 \cdot \vec{a} = \vec{a}$
9. $0 \cdot \vec{a} = \vec{0}$
10. $\vec{a} + \left(-1\right)\vec{b} = \vec{a} - \vec{b}$

If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and k is a scalar, then,

1. $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
4. $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (k\vec{b})$
5. $\vec{0} \cdot \vec{a} = 0$

If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and k is a scalar, then,

1. $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$
2. $(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b}) = \vec{a} \times (k\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

CHAPTER 5

VECTOR TRANSFORMATIONS

Now that we've talked about vectors, they're going to give us a whole new way to look at parametric equations. This is because we can always associate a position vector, a vector whose initial point is at the origin, with the corresponding point that it terminates at. For example, the point $(2,3)$ in two dimensions can be described either by saying $x = 2$ and $y = 3$, or by giving the position vector $\vec{r} = 2\hat{i} + 3\hat{j}$. Now let's apply this to something like a circle of radius 1 with center at the origin.

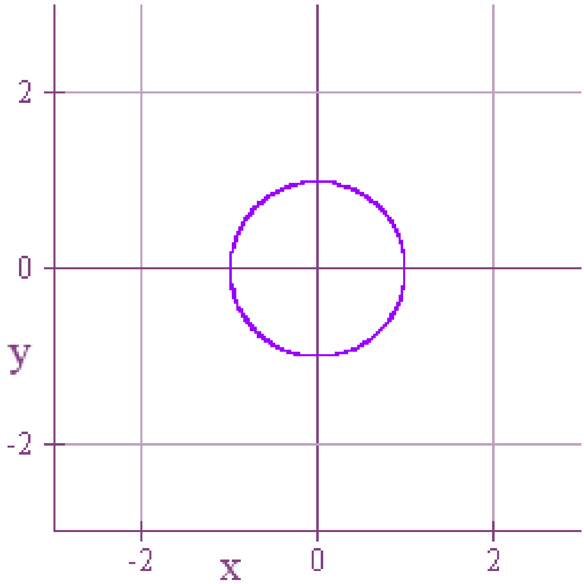
Recall that we saw previously that a good parametrization for the unit circle is,

$$\begin{aligned} &x = \cos t \\ &y = \sin t \\ &0 \le t \le 2\pi \end{aligned}$$

We can now write this as a vector-value function as,

$$\begin{aligned} &\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j} \\ &0 \le t \le 2\pi \end{aligned}$$

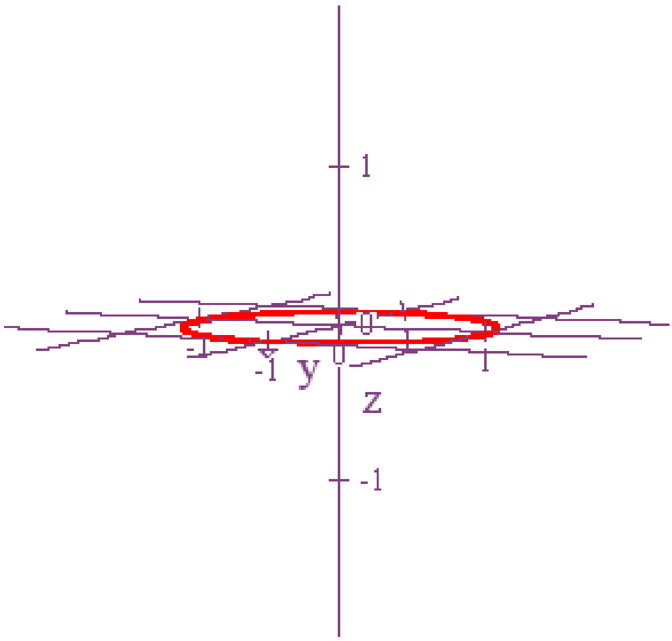
and the graph is the same circle we saw before. Also, if you want a larger circle, just multiply $\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j}$ by whatever you would like the radius to be.



If we want to now graph this unit circle in three dimensions, the vector representation again tells us how to do it. All we need to do is set the $\hat{k}$-component equal to zero.

$$\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j} + 0\hat{k}$$
$$0 \le t \le 2\pi$$

And the end result is a circle in 3-dimensional space.

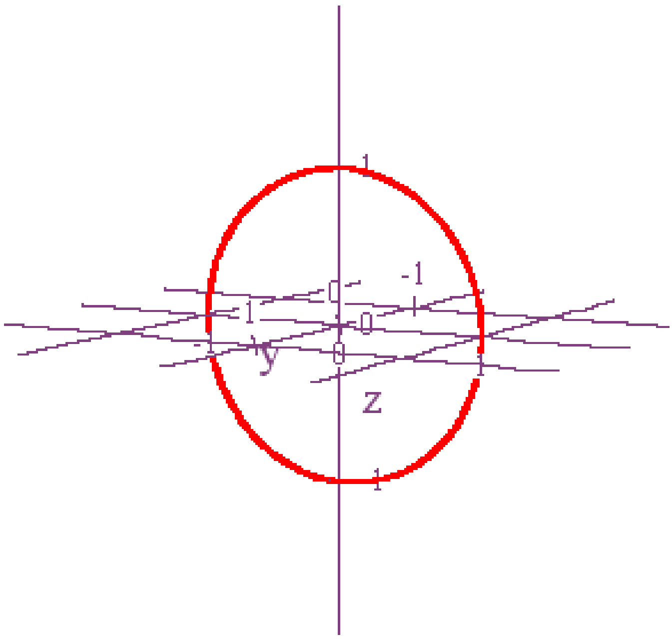


What if we now want our circle in the *yz*-plane instead of the *xy*-plane? No problem!

Just make your $\hat{i}$-component zero and move your cosine and sine functions to the $\hat{j}$

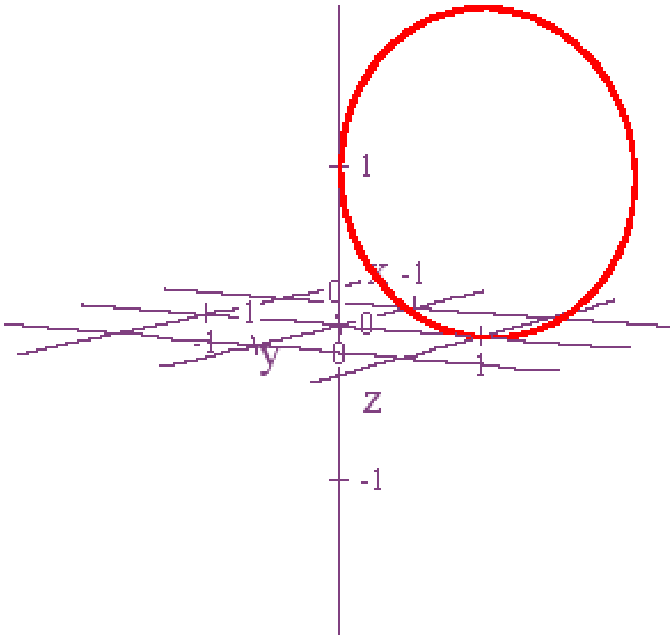
and $\hat{k}$-components.

$$\vec{r}(t) = 0\hat{i} + \cos(t)\,\hat{j} + \sin(t)\,\hat{k}$$
$$0 \le t \le 2\pi$$



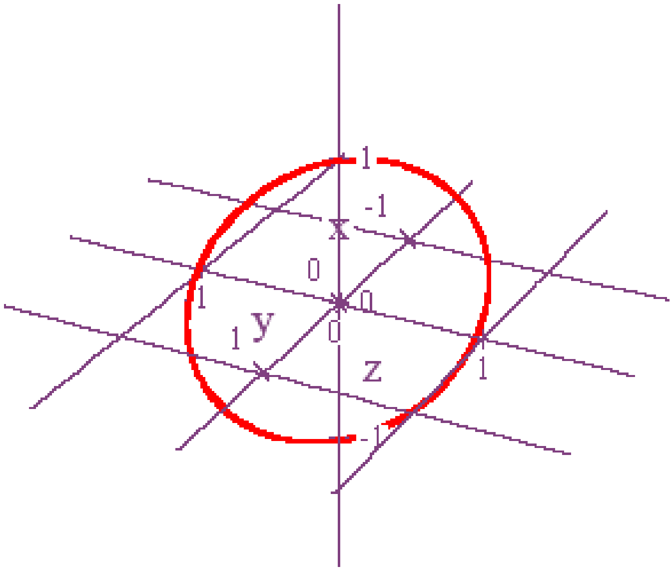
Now suppose we want to shift this circle 1 unit along the positive *y*-axis and 1 unit along the positive *z*-axis. Again, there is no difficulty if we use vectors. We just need to take the vector $\vec{u} = \hat{j} + \hat{k}$ and add it to the vector $\vec{r}$.

$$\vec{r} + \vec{u} = 0\hat{i} + (\cos(t)+1)\,\hat{j} + (\sin(t)+1)\hat{k}$$
$$0 \le t \le 2\pi$$



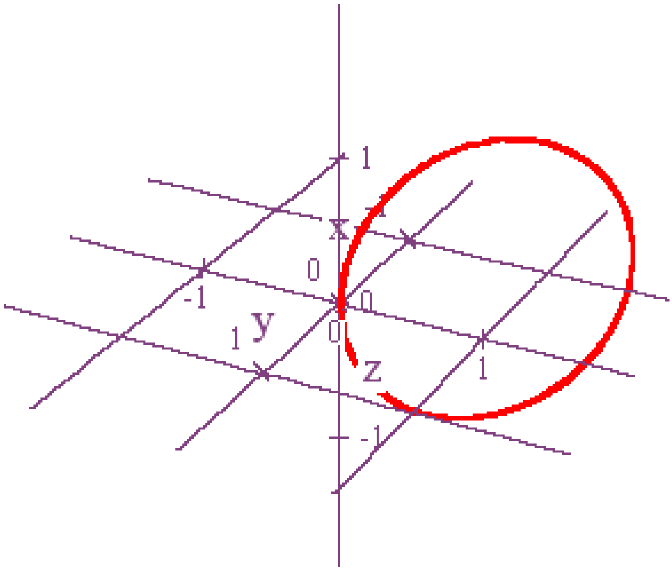
From these examples you might correctly deduce that to create a circle in any plane, all you need are two unit vectors which are perpendicular. For example, let's graph the unit circle with center at the origin that lies in the plane $y = -x$. For our unit vectors we could use $\vec{v} = \hat{k}$ and $\vec{w} = \cos\frac{3\pi}{4}\hat{i} + \sin\frac{3\pi}{4}\hat{j} = -\frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j}$. Our circle, written in vector form, will then be,

$$\vec{r}(t) = \cos(t)\vec{v} + \sin(t)\vec{w} = -\frac{\sqrt{2}}{2}\sin(t)\hat{i} + \frac{\sqrt{2}}{2}\sin(t)\hat{j} + \cos(t)\hat{k}$$
$$0 \le t \le 2\pi$$

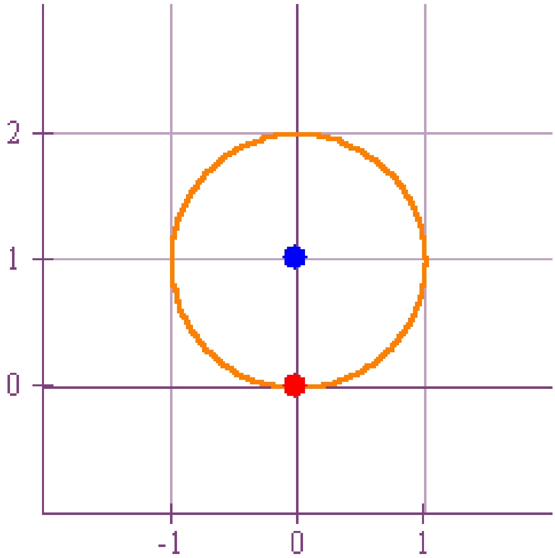


I love it when the magic works! If we now want to shift the whole thing 1 unit in the direction of the vector $\vec{w}$, then just add the unit vector $\vec{w}$ to $\vec{r}$.

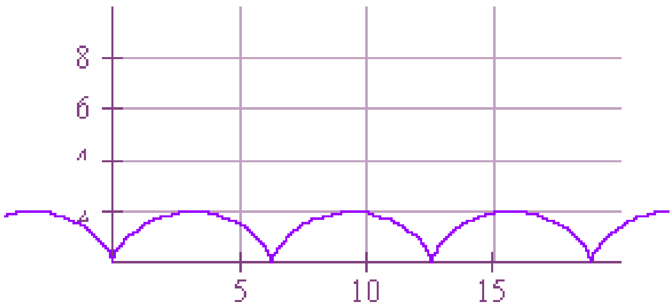
$$\vec{r}(t)+\vec{w}=\left(-\frac{\sqrt{2}}{2}\sin(t)-\frac{\sqrt{2}}{2}\right)\hat{i}+\left(\frac{\sqrt{2}}{2}\sin(t)+\frac{\sqrt{2}}{2}\right)\hat{j}+\cos(t)\hat{k}$$
$$0\le t\le 2\pi$$



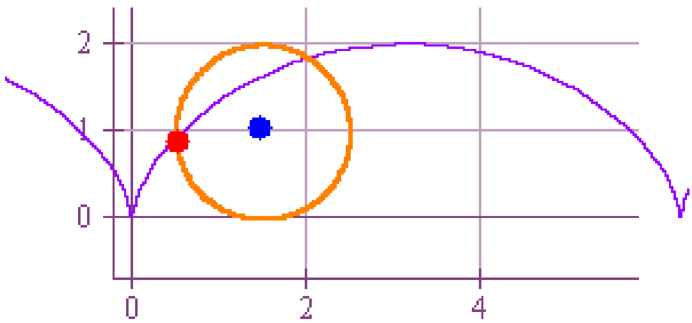
And again it worked! Now let's take something a little more challenging. Let's see if we can use a vector-valued function to come up with parametric equations for a cycloid. What's a cycloid? Well, if you are riding a bicycle in a straight line, and if you put a red dot on some particular point on your tire, then the path that red dot traces out as you go along is called a cycloid. Mathematicians thought about cycloids a lot a few hundred years ago. Personally, I don't care about them at all, but it does make for an interesting example. So let's suppose we have a circle of radius r with center at $(0,r)$, and let's also suppose that we've marked our circle with a red dot at the point $(0,0)$.



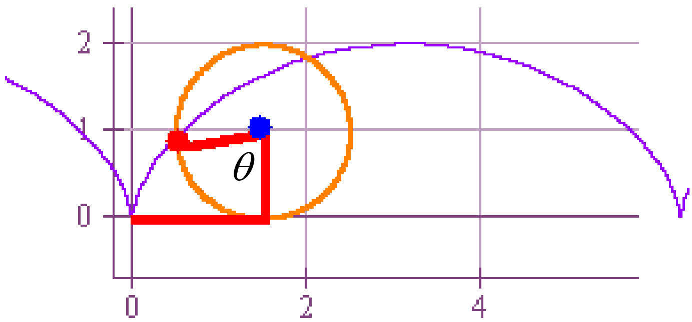
We now ask ourselves what path the red dot will trace out as we roll the circle right or left. The path that is traced is what we call a cyloid.



Now let's suppose that we've rolled our circle a little to the right. Then we might get a picture like the one below.



What we want to do is figure out the location of the red dot, and it turns out that this is not that hard to do if we describe everything in terms of vectors. To arrive at the position of the red dot, all we need to do is add three vectors together.



The first vector that we'll call $\vec{r}_1$ starts at the origin and extends horizontally to the right until it meets the circle. If you think of the circle "unwrapping" as it moves along, then the length of $\vec{r}_1$ is the same as the length of the arc of the circle corresponding to the angle θ. From trigonometry we know that the length of this arc is $r\theta$, the product of the radius and the angle. Because of the direction of movement, we set $\vec{r}_1 = r\theta\,\hat{i}$. The next vector, $\vec{r}_2$, points straight up and has length *r*. Therefore, $\vec{r}_2 = r\,\hat{j}$. The third vector, which goes from the center of the circle to a point on the circle, is a little trickier. However, imagine moving this vector so that it's initial point is at the origin. If we do that, then the angle made with the positive *x*-axis would be $\frac{3\pi}{2} - \theta$. Hence, we should define our third vector as,

$$\begin{aligned}\vec{r}_3 &= r\cos\left(\frac{3\pi}{2}-\theta\right)\hat{i} + r\sin\left(\frac{3\pi}{2}-\theta\right)\hat{j} \\ &= r\left[\cos\left(\frac{3\pi}{2}\right)\cos\left(\theta\right)+\sin\left(\frac{3\pi}{2}\right)\sin\left(\theta\right)\right]\hat{i} + r\left[\sin\left(\frac{3\pi}{2}\right)\cos\left(\theta\right)-\cos\left(\frac{3\pi}{2}\right)\sin\left(\theta\right)\right]\hat{j} \\ &= -r\sin\left(\theta\right)\hat{i} - r\cos\left(\theta\right)\hat{j}\end{aligned}$$

Hence,

$$\vec{r}(\theta) = \vec{r}_1(\theta) + \vec{r}_2(\theta) + \vec{r}_3(\theta) = \left(r\theta - r\sin(\theta)\right)\hat{i} + \left(r - r\cos(\theta)\right)\hat{j}$$

Thus, the cycloid is defined by the following parametric equations.

$$\begin{aligned}&x = r\left(\theta - \sin\theta\right) \\ &y = r\left(1-\cos\theta\right) \\ &-\infty < \theta < \infty\end{aligned}$$

Is that cool or what! In our graphs above, we used $r = 1$ for simplicity. Nonetheless, you can see how thinking terms of vectors greatly simplified the problem.

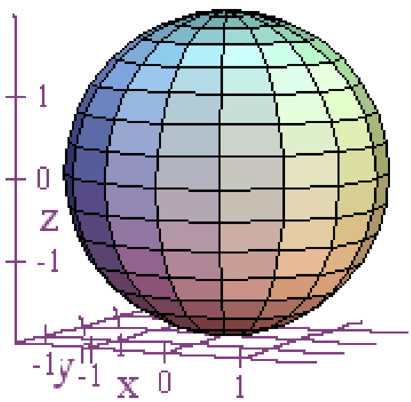
Up until now we've looked only at parametric equations or vector-valued functions of a single variable, and the result has always been a curve in space. However, there's no reason why we can't expand our explorations to equations involving two variables, and when we do so, we'll generally get surfaces instead of curves. And actually, you've already seen parametric equations involving more than one variable. Just think back to the equations we set up for converting from cylindrical and spherical coordinates back to rectangular coordinates. These were each parametric equations with two parameters. In particular, if we want to describe a sphere of radius $\rho = 2$, then our parametric equations are,

$$x = \rho \sin\varphi \cos\theta = 2\sin\varphi\cos\theta$$
$$y = \rho \sin\varphi \sin\theta = 2\sin\varphi\sin\theta$$
$$z = \rho \cos\varphi = 2\cos\varphi$$
$$0 \le \varphi \le \pi$$
$$0 \le \theta \le 2\pi$$

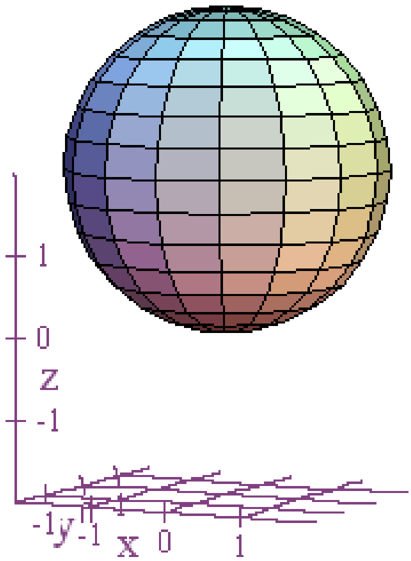
Or course, we could also write this in vector form as,

$$\vec{r}\left(\varphi,\theta\right) = \left(2\sin\varphi\cos\theta\right)\hat{i} + \left(2\sin\varphi\sin\theta\right)\hat{j} + \left(2\cos\varphi\right)\hat{k}$$
$$0 \le \varphi \le \pi$$
$$0 \le \theta \le 2\pi$$

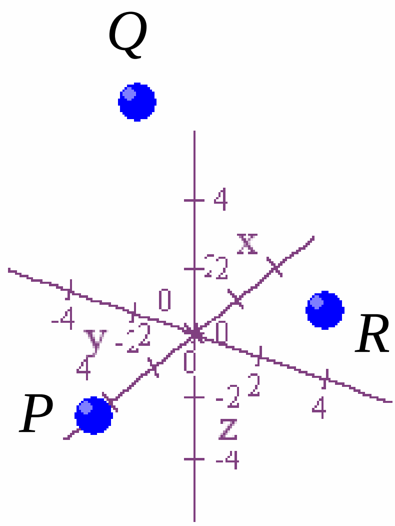
And if we graph these equations, then sure enough we get a sphere!



As we did previously, if we want to move this sphere to a different location, we just need to add the appropriate vector to it. For example, if we set $\vec{u} = 2\hat{k}$, then $\vec{r}(\varphi,\theta)+\vec{u}$ will shift our sphere 2 units upwards.

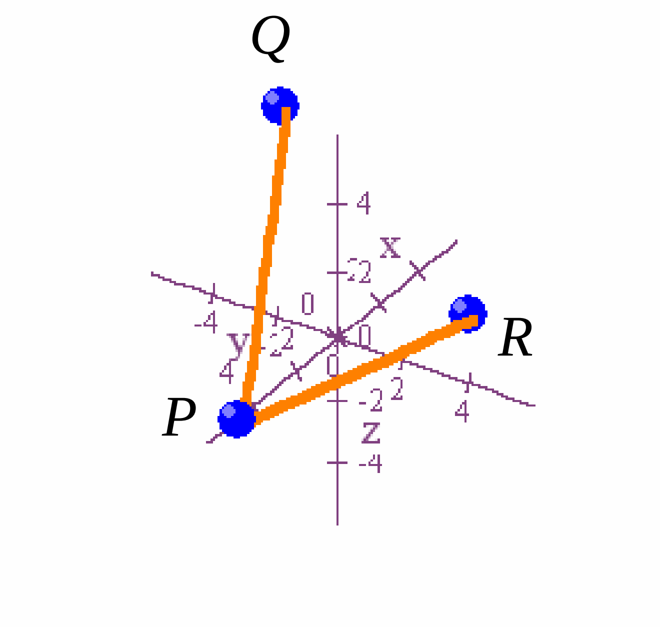


Now let's look at how we can construct a plane using parametric equations with two variables. We'll do it all from scratch starting with three distinct points $P=(6,1,1)$, $Q=(-2,-3,5)$, and $R=(0,4,2)$. If we plot these points, we get something like the following.



We can now use parametric equations to draw line segments representing the displacement vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

$\overrightarrow{PQ}$:	$\overrightarrow{PR}$:
$x = 6 - 8t$	$x = 6 - 6t$
$y = 1 - 4t$	$y = 1 + 3t$
$z = 1 + 4t$	$z = 1 + t$
$0 \le t \le 1$	$0 \le t \le 1$



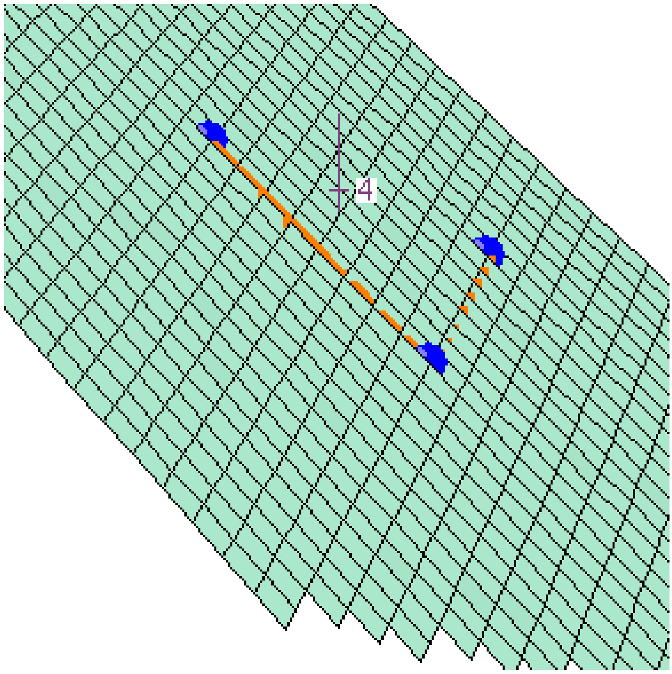
Remember, too, that if we set $t=1$, then we can write these vectors and the location of point $P=(6,1,1)$ as,

$$\vec{r}=\langle 6,1,1\rangle$$
$$\vec{u}=\langle -2,-3,5\rangle$$
$$\vec{v}=\langle 0,4,2\rangle$$

Furthermore, it should be clear from the diagram above that the vectors $\vec{u}$ and $\vec{v}$ define a plane, and we can reach any point in this plane by starting at *P* and adding on scalar multiples of $\vec{u}$ and $\vec{v}$. In other words, if we let *s* and *t* b our parameters, then $\vec{w}=\vec{r}+s\cdot\vec{u}+t\cdot\vec{v}$ defines a plane. If we write this out parametrically, we get:

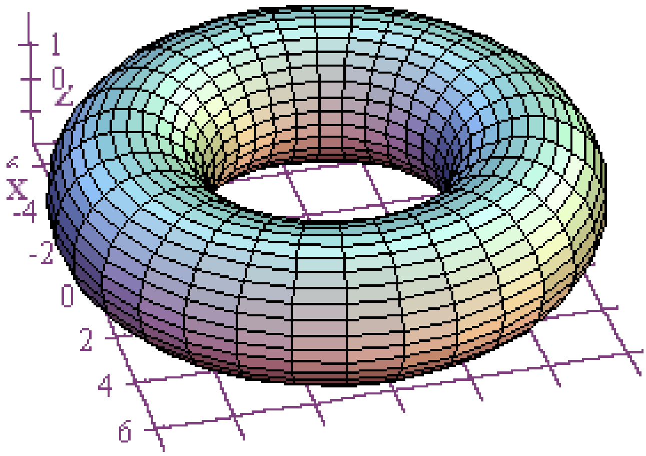
$$x=6-2s$$
$$y=1-3s+4t$$
$$z=1+5s+2t$$
$$-\infty<s<\infty$$
$$-\infty<t<\infty$$

And here's our plane!



The axes are hidden by our plane, but we can clearly see that it contains our three points and the displacement vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Thus, we now have at least a couple of ways to construct a plane using three points. We can use the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ to help us set up parametric equations in two variables for the plane, or we could get a normal vector by taking the cross product of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, and then proceeding as we did before to get an equation in the form $Ax + By + Cz + D = 0$.

Another surface worth examining is a donut shaped object called the torus. I, however, like to just refer to it as the bagel. Here's a picture of the object we're going to try to create



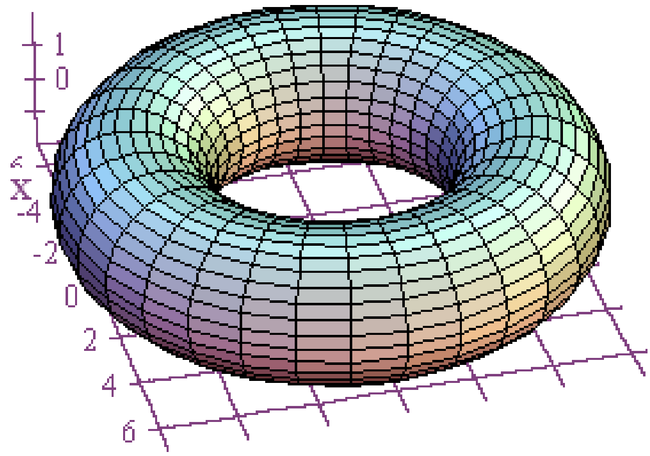
The basic idea is that we'll create parametric equations for a circle, and then rotate that circle around the *z*-axis to make the bagel. So, to begin, imagine that we fix an angle θ in the *xy*-plane that is measured with respect to the positive *x*-axis. This angle will define a plane that is orthogonal to the *xy*-plane, and we want to put our circle in this plane. To do this, recall that we need two unit vectors that are perpendicular to one another. Fortunately, it turns out that it is easy to find two such vectors. For one of the vectors we can use $\vec{u} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$. This vector is a unit vector, and it lies in both the *xy*-plane and the plane defined by our angle θ. For the second vector we can use $\vec{v} = \hat{k}$. This unit vector points straight up, and is perpendicular to $\vec{u}$. In terms of these vectors, the vector-valued function $\cos\varphi\vec{u} + \sin\varphi\vec{v},\ 0 \le \varphi \le 2\pi,$ will now define a circle of radius 1. If, however, we want our circle to be of radius 2, then we just need to multiply each component by 2 to get $2\cos\varphi\vec{u} + 2\sin\varphi\vec{v},\ 0 \le \varphi \le 2\pi.$ At this point, however, there is one small problem. The center of this circle of radius 2 is at the origin, and we need to offset it from the origin before we rotate it around the *z*-axis. Fortunately, we have a unit vector pointing in the direction we want to offset it. Namely, $\vec{u}$. So if we want to move our circle 5 units in the direction of $\vec{u}$, all we need to do is add $5\vec{u}$ to the previous result. This will give us $2\cos\varphi\vec{u} + 2\sin\varphi\vec{v} + 5\vec{u},\ 0 \le \varphi \le 2\pi.$ To now finish constructing the bagel (torus), all we need to do is rotate this around the *z*-axis by letting θ vary from 0 to 2π. Our final vector-valued function is,

$$\begin{aligned}
&\vec{r}(\theta,\varphi) = 2\cos\varphi\vec{u} + 2\sin\varphi\vec{v} + 5\vec{u} \\
&= 2\cos\varphi\cos\theta\hat{i} + 2\cos\varphi\sin\theta\,\hat{j} + 2\sin\varphi\hat{k} + 5\cos\theta\hat{i} + 5\sin\theta\,\hat{j} \\
&= \left(2\cos\varphi\cos\theta + 5\cos\theta\right)\hat{i} + \left(2\cos\varphi\sin\theta + 5\sin\theta\right)\hat{j} + 2\sin\varphi\hat{k}
\end{aligned}$$

From this, we can see that our parametric equations should be,

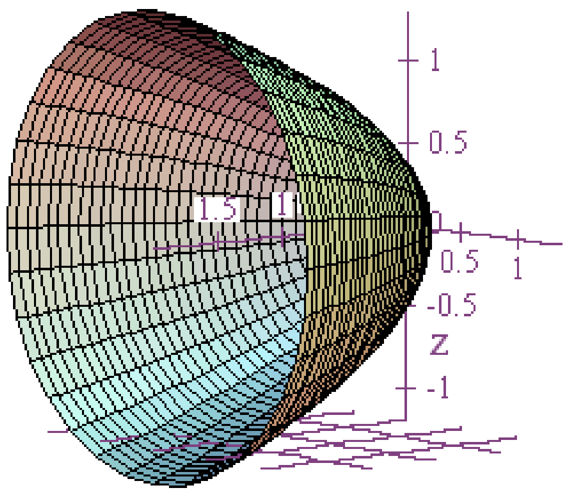
$$\begin{aligned}
&x = 2\cos\varphi\cos\theta + 5\cos\theta \\
&y = 2\cos\varphi\sin\theta + 5\sin\theta \\
&z = 2\sin\varphi \\
&0 \le \varphi \le 2\pi \\
&0 \le \theta \le 2\pi
\end{aligned}$$

Again, here is the wonderful graph that results!

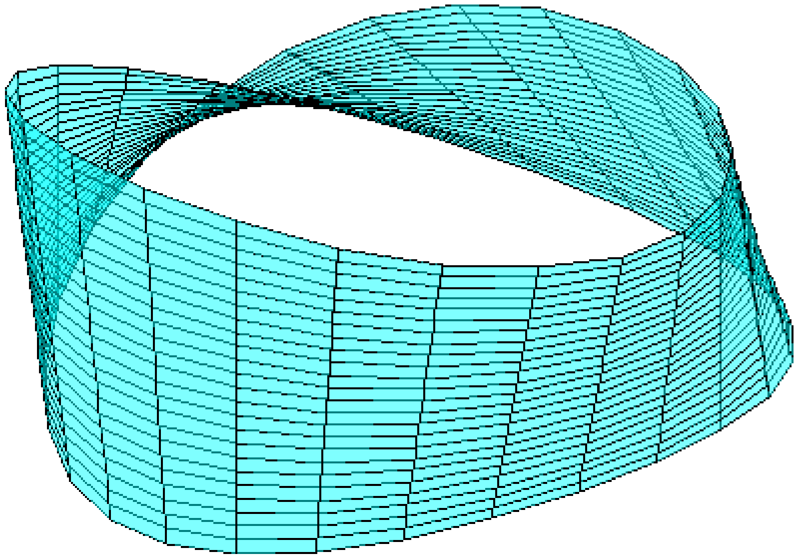


A variation of what we just did can be used to create, for example, solids of revolution about the *x*-axis. For this example, let's suppose that we want to rotate the curve $y = x^2$ about the *x*-axis. Then let's begin by expressing this curve parametrically as $x = t$, $y = \sqrt{t}$. Next, we need to create circular cross-sections that will be parallel to the *xz*-plane. Fortunately, we an use our unit vectors $\hat{j}$ and $\hat{k}$ for this purpose, and, hence, $\vec{v} = \cos\theta\,\hat{j} + \sin\theta\,\hat{k}$ will give us a unit circle parallel to the *xz*-plane. However, we don't want the radius of our circle to always be 1. We want it to be equal to our function value as we move along the *x*-axis. In particular, when $x = t$, we want the radius of our circular cross-section to be $\sqrt{t}$. Thus, let's set $\vec{v} = \sqrt{t}\cos\theta\,\hat{j} + \sqrt{t}\sin\theta\,\hat{k}$. If we now set $\vec{w} = t\,\hat{i}$, then we'll get the correct equation for the solid of revolution by simply adding $\vec{v}$ and $\vec{w}$ together. In plain English, go out *t* units on the *x*-axis, draw a circle of radius $\sqrt{t}$, and then repeat with different values of *t*. Here's the final equation and the graph that results.

$$\begin{aligned}
&\vec{r}(t,\theta) = t\,\hat{i} + \sqrt{t}\cos\theta\,\hat{j} + \sqrt{t}\sin\theta\,\hat{k} \\
&x = t \\
&y = \sqrt{t}\cos\theta \\
&z = \sqrt{t}\sin\theta \\
&0 \le \theta \le 2\pi \\
&0 \le t \le 2
\end{aligned}$$



Now let's look at a more unusual graph described by parametric equations, namely the Möbius strip. This is an object formed by taking a strip of paper, giving it a half-twist, and then connecting the ends. The resulting surface looks like this.



This is an example in mathematics of what we call a one-sided surface. In other words, if you trace a line lengthwise along this object, then you'll see when done that you've traced the line on both of what you may have thought were two different sides. A better way to convince yourself that this surface is different is to cut it in half lengthwise along the line you traced. When you finish this, you'll discover that you have just one long twisted piece instead of two pieces. If nothing else, this should convince you that it doesn't have two sides. Otherwise, when we cut it in half we should get two pieces! The parametric equations for the above Möbius strip are as follows,

$$x = \cos u + v\sin\left(\frac{u}{2}\right)\cos u$$

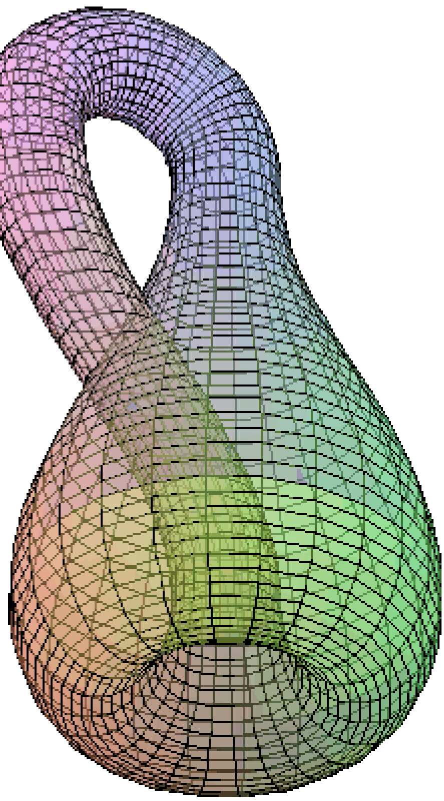
$$y = \sin u + v\sin\left(\frac{u}{2}\right)\cos u$$

$$z = v\cos\left(\frac{u}{2}\right)$$

$$0 \le u \le 2\pi$$

$$-0.5 \le v \le 0.5$$

An even more intriguing object is the Klein bottle which is pictured below.



Technically, this is a surface that takes a twist through a fourth dimension in order to create an object whose inside is the same as its outside. Also, while this may seem a little fanciful, isn’t that the way we are? In other words, we normally tend to think of ourselves as a ghost in a machine whose thoughts and perceptions occur on the inside in response to what goes on on the outside. However, every time we have a thought or emotion, doesn’t that register as brain activity on the outside, in the physical world? In this sense, we can’t really make a clear distinction between our inside and our outside, and that is why we are like living Klein bottles. Everything that occurs “inside” us has its corresponding marker “outside” of us back in the physical world. By the way, mathematicians refer to both the Möbius strip and the Klein bottle as nonorientable figures. In other words, we can’t give either object any sort of orientation with respect to side. In each object, what appear to be separate sides are really the same. With most things we encounter in life, we can assign some clear orientation like north, south, east, or west, or democrat or republican, but with these objects, the surfaces lack the appropriate orientations that are needed to create separation between sides. Consequently, I often think that in order to transcend separation and become one with anything, you first need to become a little disoriented!

To generate the nice image of the Klein bottle above, I had to use four sets of parametric equations and then combine the images when done. For future reference, here are my equations.

$$Klein1:$$
$$x = (2.5 + 1.5\cos v)\cos u$$
$$y = (2.5 + 1.5\cos v)\sin u$$
$$z = 3v$$
$$0 \le u \le 2\pi$$
$$0 \le v \le \pi$$

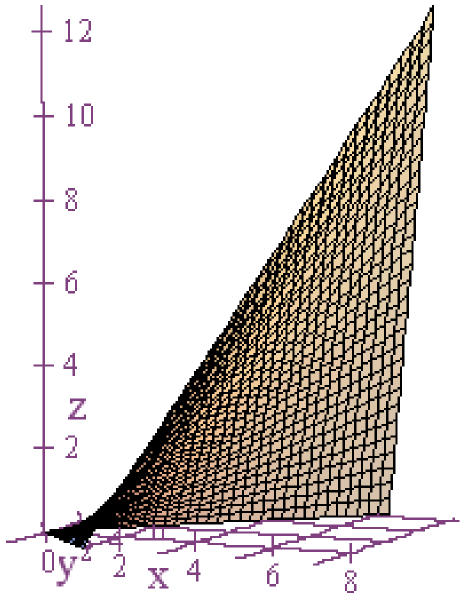
$$Klein2:$$
$$x = 2 - 2\cos v + \sin u$$
$$y = \cos u$$
$$z = 3v$$
$$0 \le u \le \pi$$
$$0 \le v \le \pi$$

$$Klein3:$$
$$x = 2 + (2 + \cos u)\cos v$$
$$y = \sin u$$
$$z = 3\pi + (2 + \cos u)\sin v$$
$$0 \le u \le \pi$$
$$0 \le v \le \pi$$

$$Klein4:$$
$$x = (2.5 + 1.5\cos v)\cos u$$
$$y = (2.5 + 1.5\cos v)\sin u$$
$$z = -2.5\sin v$$
$$0 \le u \le \pi$$
$$0 \le v \le \pi$$

Now let's look at something quite different. We'll start, though, with some parametric equations that look very innocuous at first glance.

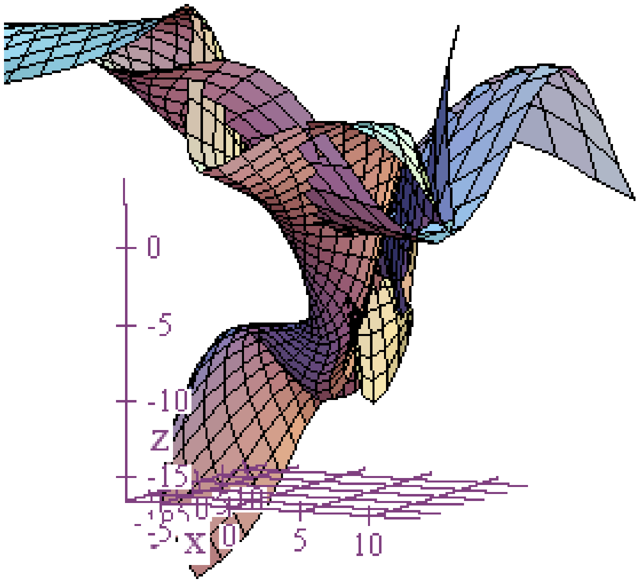
$$x = s + t$$
$$u = s - t$$
$$z = st$$



Now, however, suppose we scramble everything up by putting our variables in a row matrix, $\begin{pmatrix} s+t & s-t & st \end{pmatrix}$, and multiplying on the right by the matrix

$$\begin{pmatrix} \cos(t+t) & -\sin(t) & \cos(st) \\ \sin(s+t) & 5 & -3 \\ 2 & \sin(s) & \cos(s-t \end{pmatrix}, \text{ where } -3 \le s \le 3 \text{ and } -3 \le t \le 3.$$

Then the resulting graph looks like this. I call this one my Klingon bird of prey!

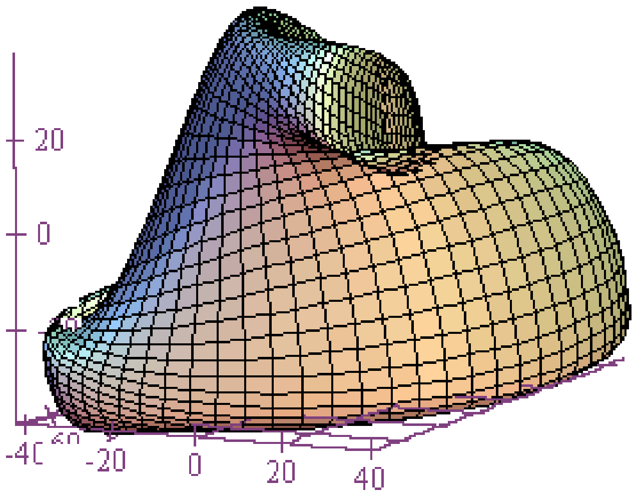


We can create a lot of interesting graphs by taking some standard parametric equations in two variables, and then multiplying on the right by a matrix to scramble them up. Then you can add another twist to the mix by deciding whether you want to treat the final result as rectangular, cylindrical, or spherical coordinates. This is where it really gets bizarre! Here are some more examples for you to ponder.

1. $$(x,y,z)_{\text{rectangular}} = \left(\left[10+4\cos s\right]\cos t, \left[10+4\cos s\right]\sin t, 4\sin s\right)\begin{pmatrix} 3 & 5\sin s & \cos 3s \\ \sin s & 5 & \sin s \\ 2\cos s & -\sin s & -8 \end{pmatrix}$$

$0 \le s \le 2\pi$

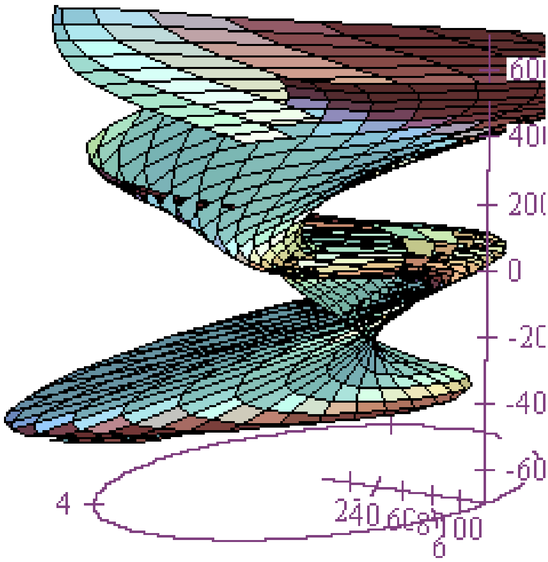
$0 \le t \le 2\pi$



2. $(r,\theta,z)_{\text{cylindrical}} = \left(s+t, s-t, s^2-t^2\right)\begin{pmatrix} 3 & 5\sin s & \cos 3s \\ \sin s & 5 & \sin s \\ 2\cos s & -\sin s & -8 \end{pmatrix}$

$0 \le s \le 3\pi$

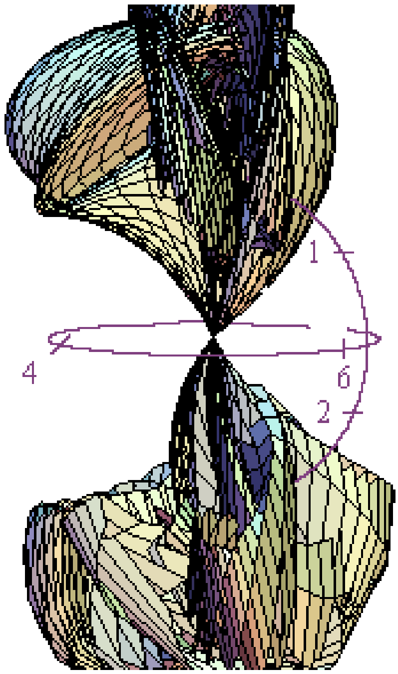
$0 \le t \le 3\pi$



3. $(\rho,\theta,\varphi)_{\text{spherical}} = \left(\cos(s+t),\sin(st),\cos(s-t)\right)\begin{pmatrix} 3 & 5\sin s & \cos 3s \\ \sin s & 5 & \sin s \\ 2\cos s & -\sin s & -8 \end{pmatrix}$

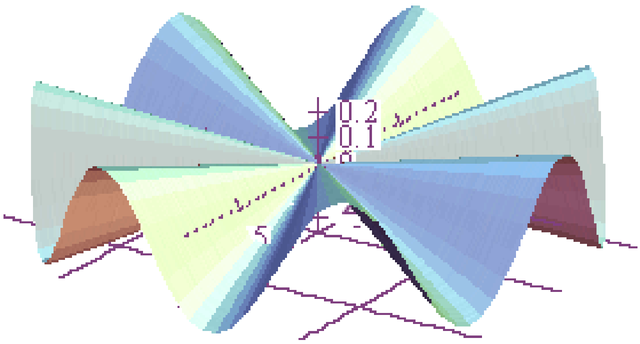
$0 \le s \le 2\pi$

$0 \le t \le 2\pi$

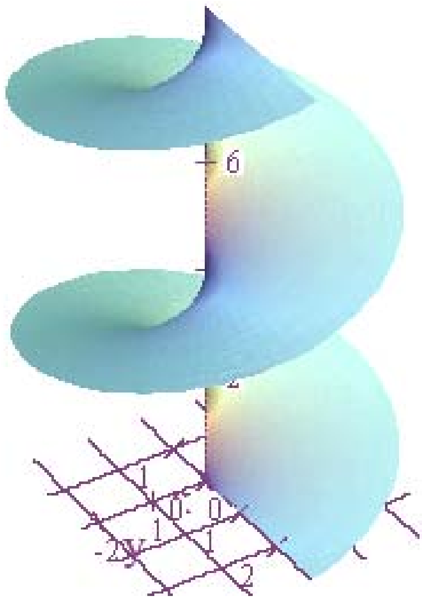


A now we'll close this chapter on vector transformations and parametric equations in two variables by presenting a variety of interesting graphs that mathematicians have discovered. Enjoy!

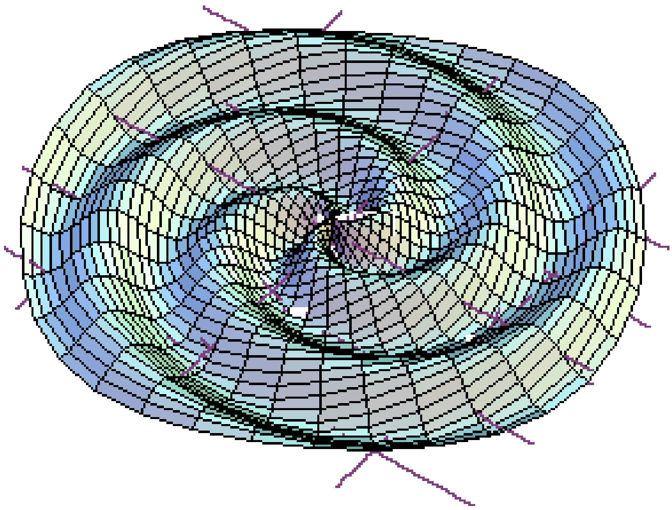
4. $x = t\cos s$
$y = t\sin s$
$z = \dfrac{t\cos 7s}{4}$
$0 \le s \le 2\pi$
$0 \le t \le 1$



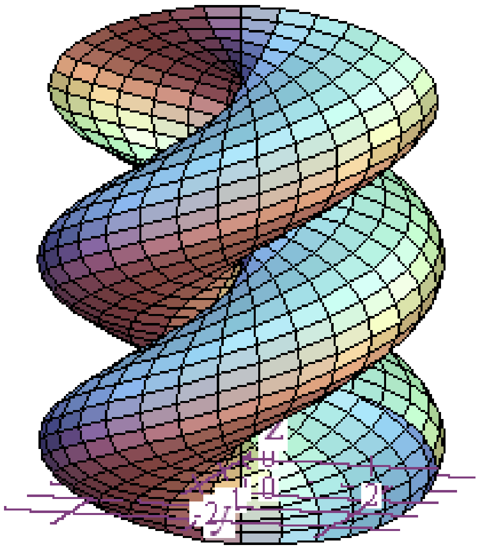
5. $x = t\cos s$
$y = t\sin s$
$z = 0.7s$
$0 \le s \le 4\pi$
$0 \le t \le 3$



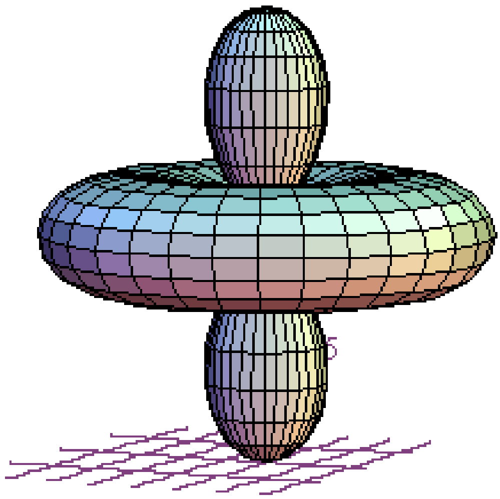
6. $x = t\cos s$
$y = t\sin s$
$z = \cos(-3s + t)$
$0 \le s \le 2\pi$
$0 \le t \le 5\pi$



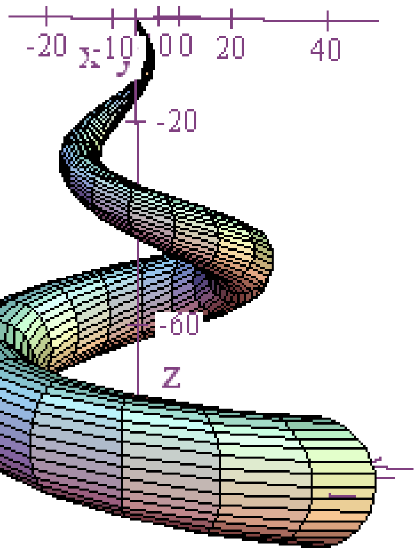
7. $x = 3\cos s \cos t$
$y = 3\sin s \cos t$
$z = 3\sin t + 2s$
$0 \le s \le 2\pi$
$-\pi \le t \le \pi$



8. $x = \cos 2s \sin s \cos t$
$y = \cos 2s \sin s \sin t$
$z = \cos 2s \cos s$
$0 \le s \le \pi$
$0 \le t \le 2\pi$



9. $x = s\cos s(3+\cos t)$
$y = s\sin s(3+\cos t)$
$z = s\sin t - 7s$
$0 \le s \le 4\pi$
$0 \le t \le 2\pi$



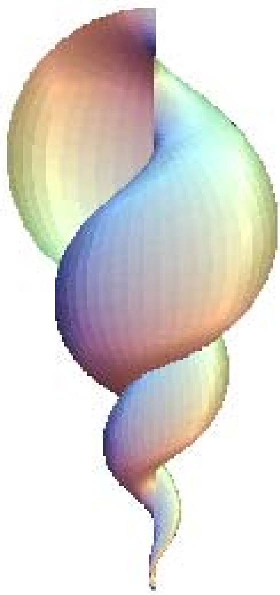
10. $x = \frac{t}{18}\sin s \cos t$

$y = \frac{t}{18}\sin s \sin t$

$z = \frac{t}{18}\cos s + \frac{t}{6}$

$0 \le s \le \pi$

$0 \le t \le 6\pi$



CHAPTER 6

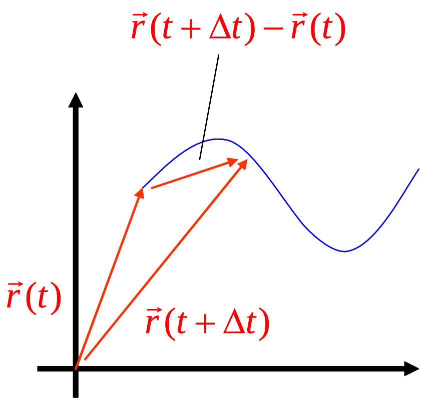
VECTOR CALCULUS

We've spent a lot of time so far just looking at all the different ways you can graph things and describe things in three dimensions, and it certainly seems like there is a lot more variety than what we encounter in just two dimensions. So much so that, in case you haven't noticed, we haven't done a single integral or derivative yet! Well, maybe we can remedy that in this chapter. Maybe it's just about time to see what we can do with derivatives and integrals when we apply them to vector-valued functions. In fact, for starters, let's take a look at the standard parametrization for the unit circle, $\vec{r}(t) = \cos(t)\,\hat{i} + \sin(t)\,\hat{j}$, where $0 \le t \le 2\pi$. If we want to find the derivative of this function, we have to go back to the basics, the way in which a derivative was defined for you in your first calculus course. Thus, recall that one of the definitions you may have been given was $\dfrac{df}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$. If we try something similar with the function $\vec{r}(t) = \cos(t)\,\hat{i} + \sin(t)\,\hat{j}$, then we get

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\left[\cos(t+\Delta t)\,\hat{i} + \sin(t+\Delta t)\,\hat{j}\right] - \left[\cos(t)\,\hat{i} + \sin(t)\,\hat{j}\right]}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{\left[\cos(t+\Delta t) - \cos(t)\right]\hat{i} + \left[\sin(t+\Delta t) - \sin(t)\right]\hat{j}}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{\cos(t+\Delta t) - \cos(t)}{\Delta t}\,\hat{i} + \frac{\sin(t+\Delta t) - \sin(t)}{\Delta t}\,\hat{j}$$
$$= \frac{d(\cos t)}{dt}\,\hat{i} + \frac{d(\sin t)}{dt}\,\hat{j} = -\sin(t)\,\hat{i} + \cos(t)\,\hat{j}$$

In other words (and more generally), if we have a function $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$,

then $\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$. We differentiate by simply differentiating each component function, and when we do, we get another vector! Graphically, things look like this.



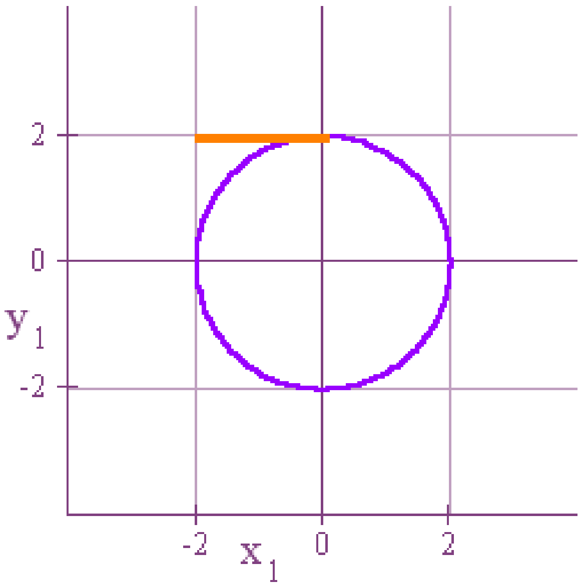
$$\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

As Δt gets smaller, our quotient $\frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$ gets closer and closer to being tangent to our curve. Thus, if we evaluate $\vec{r}\,'(t)$ at a particular value of *t*, then we get a vector which is tangent to the curve $\vec{r}(t)$ at the point corresponding to that *t* value.

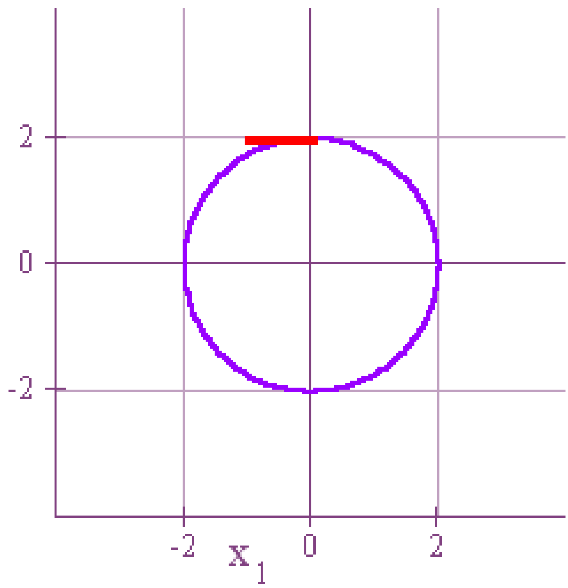
Now let's do a drawing.

We're going to start over with the function $\vec{r}(t) = 2\cos(t)\hat{i} + 2\sin(t)\hat{j}$, where $0 \le t \le 2\pi$.

This is also a circle with center at the origin, but this time the radius is 2. Also, it's easy to see that $\frac{d\vec{r}}{dt} = -2\sin(t)\hat{i} + 2\cos(t)\hat{j}$. Furthermore, if we evaluate this derivative at $t=\frac{\pi}{2}$, we get $\left.\frac{d\vec{r}}{dt}\right|_{t=\pi/2}=-2\sin(\pi/2)\,\hat{i}+2\cos(\pi/2)\,\hat{j}=-2\hat{i}$. The corresponding point on our circle when $t=\frac{\pi}{2}$ is $(0,2)$. Thus, if we set $x=-2t$ and $y=2$ with $0\le t\le 1$, then we can add the graph of this tangent vector to the graph of our vector-valued function. And here it is!

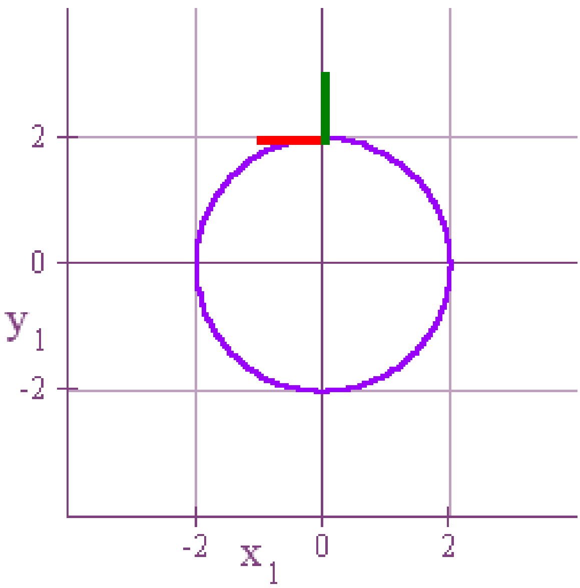


Out tangent vector points in the direction it does because with our particular parametrization, we are tracing the circle in the counterclockwise direction. Also, notice that the length of our tangent vector $\left.\frac{d\vec{r}}{dt}\right|_{t=\pi/2}=-2\sin(\pi/2)\,\hat{i}+2\cos(\pi/2)\,\hat{j}=-2\hat{i}$ is 2. However, for many applications that will come later, we will want to have not just any ol' tangent vector, but a tangent vector of length 1, what we call the unit tangent vector. To get a formula for the unit tangent, just divide the derivative of our vector-valued function by its length. Thus, unit tangent $= T = \dfrac{\frac{d\vec{r}}{dt}}{\left\| \frac{d\vec{r}}{dt} \right\|} = \dfrac{\vec{r}'(t)}{\left\| \vec{r}'(t) \right\|}$. In the case before us, $T = -\hat{i}$, and a parametrization for this starting at $(0,2)$ is $x = -t$, $y = 2$, and $0 \le t \le 1$. And here's the new picture.



As you can see, then length has been adjusted from 2 to 1.

In addition to the unit tangent vector, we can also define a unit normal vector which is perpendicular to the unit tangent. If we are in 3-dimensional space, the calculations can become a little complicated because at any given point on a circle, there are an infinite number of unit vectors that are perpendicular to the unit tangent. Just imagine taking on such vector at a point and then rotating it full circle about the unit tangent, and you'll see how we wind up with an infinite number. We have an infinite number of directions such a normal vector could point in. However, in two dimensions things are much simpler. For example, in the diagram above there are only two directions the unit normal could point in when placed at the designated point. It could point either straight up or straight down, and that's it! Consequently, things are simpler, and once we've found a unit tangent vector, the procedure for getting the unit normal will be very simple indeed. And here's how we do it. Suppose that at some point on a curve in two dimensions that the unit tangent vector is $T = S\hat{i} + R\hat{j}$. Then the unit normal vector is $N = R\hat{i} - S\hat{j}$. We just switch the components *S* and *R*, and then we add a minus sign in front of the *S*. And how do we know that the result is a unit vector? Because since *T* is a unit vector, it's length is $\|T\| = \sqrt{S^2 + R^2} = 1$, and thus, $\|N\| = \sqrt{R^2 + (-S)^2} = 1$. And how do we know that *N* is perpendicular to *T*? Because the dot product $T \bullet N = SR - RS = 0$. For our diagram above, $T = -\hat{i}$, and so $N = \hat{j}$.



By the way, there is a simple way to understand the direction of both the unit tangent and the unit normal vector. First, it should be clear that the unit tangent vector basically points in the direction that corresponds to the direction in which you are tracing out your curve. Because the circle above comes into being in a counterclockwise direction as we let our angle increase from 0 to 2π, our unit tangent takes its direction from that orientation. If we were tracing our curve in the clockwise direction, then the unit tangent would likewise point in the opposite direction. Once we understand the direction of our unit tangent, then it's very easy to know which direction the unit normal will point in. Just imagine this. If you are facing the direction in which the unit tangent is pointing, then the unit normal will always be at right angles to it pointing to your right. This is a consequence of the way in which we defined the unit tangent, and if you graph a few examples by hand on paper, then you'll understand why our unit normal is always the one that points to the right of the unit tangent.

Now let's repeat everything with another point on the circle. In particular, let's let $t = \frac{\pi}{4}$. Then,

$$\vec{r}(\pi/4) = 2\cos(\pi/4)\hat{i} + 2\sin(\pi/4)\,\hat{j} = \sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j}$$
$$\vec{r}'(\pi/4) = -2\sin(\pi/4)\hat{i} + 2\cos(\pi/4)\,\hat{j} = -\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j}$$
$$\|\vec{r}'(\pi/4)\| = \sqrt{2+2} = \sqrt{4} = 2$$
$$T(\pi/4) = \frac{\vec{r}'(\pi/4)}{\|\vec{r}'(\pi/4)\|} = \frac{-1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$
$$N(\pi/4) = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

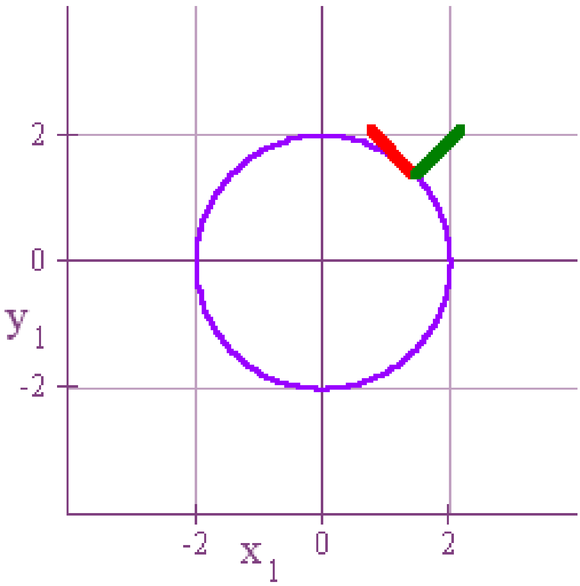
Well that's all we need to know in order to set up parametric equations for graphing these vectors. For the unit tangent, we use,

$$x = \sqrt{2} - \frac{1}{\sqrt{2}}t$$
$$y = \sqrt{2} + \frac{1}{\sqrt{2}}t$$
$$0 \le t \le 1$$

And for the unit normal we us,

$$x = \sqrt{2} + \frac{1}{\sqrt{2}} t$$
$$y = \sqrt{2} + \frac{1}{\sqrt{2}} t$$
$$0 \le t \le 1$$

And here's the result!



In many situations, it is common to think of our parameter *t* as representing time, and $\vec{r}(t)$ as giving the position of an object at time *t*. In this setup, we already have a word for the rate of change of position with respect to time. That's what we call velocity. Thus, we will often designate the first derivative of our vector-valued function $\vec{r}(t)$ as a velocity vector $\vec{v}(t)$. In other words, $\vec{r}'(t) = \frac{d\vec{r}}{dt} = \vec{v}(t) = velocity$.

Furthermore, we define speed as the length of our velocity vector,

$speed = \|\vec{r}\,'(t)\| = \left\|\frac{d\vec{r}}{dt}\right\| = \|\vec{v}(t)\|$. In ordinary language and even in some other mathematics courses we often use the terms velocity and speed to mean the same thing. However, once we reach this level, we make a distinction between the two. Velocity is a vector quantity with both magnitude and direction, and speed is simply the scalar length of our velocity vector. For our circle above,

$\vec{r}(t) = 2\cos(t)\,\hat{i} + 2\sin(t)\,\hat{j}$ and $\vec{v}(t) = -2\sin(t)\,\hat{i} + 2\cos(t)\,\hat{j}$. When $t = \frac{\pi}{4}$, this becomes

$\vec{v}(\pi/4) = -2\sin(\pi/4)\,\hat{i} + 2\cos(\pi/4)\,\hat{j} = -\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j}$ and the corresponding speed is

$\left\|-\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j}\right\| = \sqrt{2+2} = \sqrt{4} = 2$.

Another thing we might look at now is the rate of change of velocity with respect to time, and we also already have a name for this. We call it acceleration. Thus,

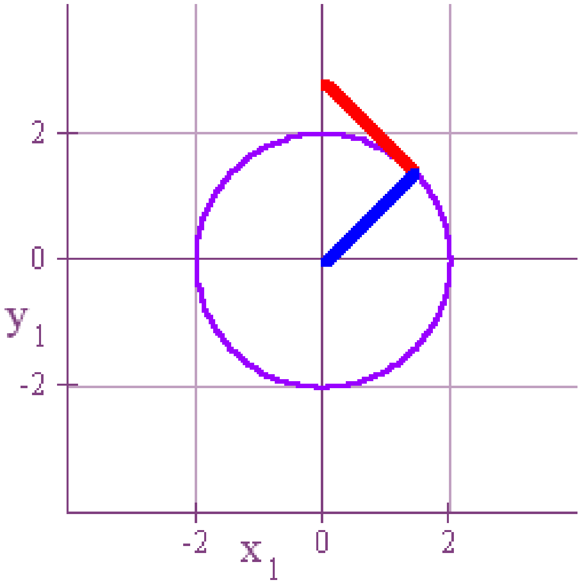
$\vec{r}\,''(t) = \vec{v}\,'(t) = \frac{d\vec{v}}{dt} = \vec{a}(t) = acceleration$. For our circle example we have

$\vec{a}(t) = \frac{d\vec{v}}{dt} = -2\cos(t)\,\hat{i} - 2\sin(t)\,\hat{j}$, and at $t = \frac{\pi}{4}$ we get

$\vec{a}(\pi/4) = -2\cos(\pi/4)\,\hat{i} - 2\sin(\pi/4)\,\hat{j} = -\sqrt{2}\,\hat{i} - \sqrt{2}\,\hat{j}$. The parametric equations we'll use for graphing line segments for these vectors are as follows,

Velocity	Acceleration
$x = \sqrt{2} - \sqrt{2}t$	$x = \sqrt{2} - \sqrt{2}t$
$y = \sqrt{2} + \sqrt{2}t$	$y = \sqrt{2} - \sqrt{2}t$
$0 \le t \le 1$	$0 \le t \le 1$

And here is our graph.



We can see in this case that the acceleration is directly towards the center of the circle as we orbit it in the counterclockwise direction. By the way, there is also a name for the derivative of acceleration. It's called the jerk! That makes sense because if an acceleration is not smooth, then the ride feels jerky. However, we won't mess with any jerks in this course.

Once we understand differentiation of vector-valued functions, we can just as easily integrate them. In particular, since differentiation is done individually for each separate component, integration must be performed in a similar manner. Thus,

$$\int_a^b \vec{r}(t)\,dt = \int_a^b x(t)dt\,\hat{i} + \int_a^b y(t)dt\,\hat{j} + \int_a^b z(t)dt\,\hat{k}\text{, and}$$

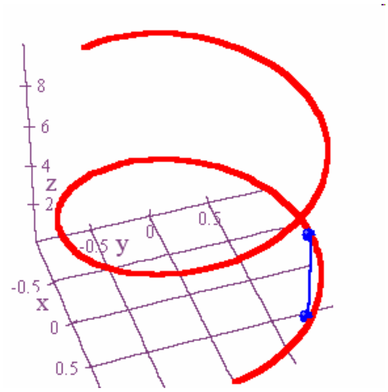
$$\int \vec{r}(t)\,dt = \int x(t)dt\,\hat{i} + \int y(t)dt\,\hat{j} + \int z(t)dt\,\hat{k} + \vec{C}$$

Notice that in the indefinite integrals you will pick up arbitrary constants of integration in each component. The sum of these arbitrary constants results in the constant vector $\vec{C}$ that we've added on at the end. Now let's take an example!

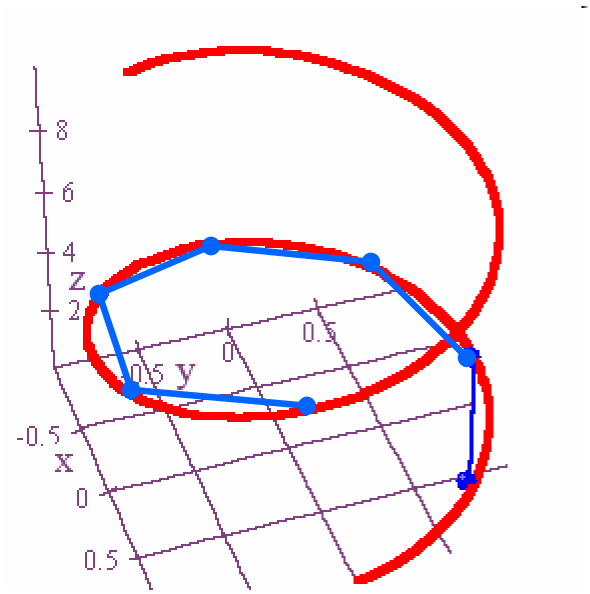
Let's suppose that our velocity vector is $\vec{v}(t) = -2\sin(t)\,\hat{i} + 2\cos(t)\,\hat{j}$ and that when $t = 0$, $\vec{r}(0) = 3\hat{i} + 2\hat{j}$. We now want to find $\vec{r}(t)$, and just as you have done in other courses, it all boils down to integrating one function and then using your condition to evaluate the constant. Thus, $\vec{r}(t) = \int \vec{v}(t)dt = 2\cos(t)\,\hat{i} + 2\sin(t)\,\hat{j} + \vec{C}.$ Now using the given condition we get $\vec{r}(0) = 2\cos(0)\,\hat{i} + 2\sin(0)\,\hat{j} + \vec{C} = 2\hat{i} + \vec{C} = 3\hat{i} + 2\hat{j} \Rightarrow \vec{C} = \hat{i} + 2\hat{j}$.

Thus, $\vec{r}(t) = 2\cos(t)\,\hat{i} + 2\sin(t)\,\hat{j} + \left(\hat{i} + 2\hat{j}\right) = \left(2\cos(t) + 1\right)\hat{i} + \left(2\sin(t) + 2\right)\hat{j}$. Notice that the graph of our end result would be a circle of radius 2 that has had its center shifted from the origin exactly one unit to the right and 2 units up, i.e. to the point $\left(1, 2\right)$.

Now let's think about another problem involving curves created by vector-valued functions. In particular, how do we find the length, also called the arc length, of our curve as our parameter varies from *a* to *b*? Let's look at a diagram to get some inspiration.



The curve above is the graph of $\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j} + t\hat{k}$ with $0 \le t \le 10$. We have two points marked on the curve in blue and a straight line is connecting the two points. We now ask ourselves what the length of the arc is from the first point to the second point. Well, if our points are close enough together, then we can effectively approximate the true arc length by simply finding the straight line distance between the points. And we know how to do the latter because our distance formula tells us that the distance between the two points will be the square root of the sum of the change in *x* squared plus the change in *y* squared plus the change in *z* squared. In other words, $\text{distance} = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$. Now let's go back and take several points off of our curve and approximate the arc lengths between them in the same way.



The result now is that for the curve overall, $Arc\,Length \approx \sum \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$. Look now at how we can rewrite this formula.

$$\sum \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} = \sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2} \cdot \Delta t$$

And finally, we can generate a formula for the exact arc length just by taking the limit of this last expression as Δt goes to 0.

$$Arc\,Length = \lim_{\Delta t \to 0} \sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2} \cdot \Delta t$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b \left\|\vec{r}'(t)\right\| dt = \int_a^b \left\|\vec{v}(t)\right\| dt$$

This last bit is an unexpected surprise. What it tells us, though, is that if we have our curve defined parametrically by a vector-valued function, then the arc length of the curve as *t* varies from *a* to *b* is equal to the integral from *a* to *b* of the norm of the velocity function. It works out this way, of course, because if

$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, then $\vec{v}(t) = \dfrac{dx}{dt}\hat{i} + \dfrac{dy}{dt}\hat{j} + \dfrac{dz}{dt}\hat{k}$, and

$\|\vec{v}(t)\| = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2}$. Applying this to a circle of radius *r* in two

dimensions with center at the origin, we could take $\vec{r}(t) = r\cos(t)\hat{i} + r\sin(t)\hat{j}$ where

$0 \le t \le 2\pi$. Thus,

$$x = r\cos(t) \quad \frac{dx}{dt} = -r\sin(t)$$
$$y = r\sin(t) \quad \frac{dy}{dt} = r\cos(t)$$

From this we get that $\sqrt{\left(\dfrac{\Delta x}{\Delta t}\right)^2 + \left(\dfrac{\Delta y}{\Delta t}\right)^2} = \sqrt{r^2\sin^2(t) + r^2\cos^2(t)} = \sqrt{r^2} = r$.

Consequently, the circumference of a circle is,

$$Arc\,Length = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{2\pi} r\,dt = rt\Big|_0^{2\pi} = 2\pi r$$

Of course, you already knew that the circumference was 2π times the radius, but it’s nice to know how to actually (and easily!) prove that fact.

Now let's look at something else that comes out of this formula for arc length. We can make the upper integration limit of our arc length integral a variable so that we can define a function that returns to us the arc length of our curve from a starting value *a* to a stopping value *t*. When we do this, we'll have to replace the variable *t* in the integrand by a dummy variable so that we're not using *t* in two different ways. Once we've done that, this is what we get.

$$s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

And now what is nice about this formula is that the Fundamental Theorem of Calculus tells us exactly how to differentiate it with respect to *t*. According to our Fundamental Theorem,

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \left\|\vec{r}'(t)\right\| = \left\|\vec{v}(t)\right\| = speed$$

Again, the fact that the derivative is equal to the speed might be totally unexpected, but when we think about it makes sense. If $s(t)$ is measuring the scalar distance traveled as a function of time, then of course the rate at which this distance changes with respect to time is nothing other than the speed at which we are traveling, i.e.

$$\frac{ds}{dt} = \left\|\vec{v}(t)\right\| = speed$$

Now let's look at one more calculus application, how to measure how curved a curve actually is. The definition that we are going to use for curvature is,

$$\kappa = \left\| \frac{dT}{ds} \right\|$$

The letter on the left is the Greek letter *kappa*, and the expression on the right is the norm of the rate of change of the unit tangent vector with respect to arc length. Let's think about this for a moment. As our unit tangent vector moves along a curve, then one thing that is not going to change is its length. In fact, the only thing that can change is its direction. If its direction changes a lot with regard to a small change in arc length, then we've got some curvature. On the other hand, if a small change in arc length produces only a small change in direction, then we have very little curvature. Thus, it makes perfectly good sense to measure curvature as the magnitude of the rate of change of the unit tangent vector with respect to arc length. Now, however, the only problem is how do we compute this? This is where the chain rule helps us! According to the chain rule, we should have that

$$\frac{dT}{dt} = \frac{dT}{ds} \cdot \frac{ds}{dt} = \frac{dT}{ds} \cdot \left\| \vec{r}\,'(t) \right\| = \frac{dT}{ds} \cdot \left\| \frac{d\vec{r}}{dt} \right\|$$

Hence, do a little division and you get,

$$\frac{dT}{ds} = \frac{\dfrac{dT}{dt}}{\left\| \dfrac{d\vec{r}}{dt} \right\|} \Rightarrow \kappa = \left\| \frac{dT}{ds} \right\| = \frac{\left\| \dfrac{dT}{dt} \right\|}{\left\| \dfrac{d\vec{r}}{dt} \right\|}$$

This gives us a formula we can use for computations! For example, if a parametrization for a circle of radius *r* with center at the origin is $\vec{r}(t) = r\cos(t)\hat{i} + r\sin(t)\hat{j}$ with $0 \le t \le 2\pi$, then $\vec{r}'(t) = -r\sin(t)\hat{i} + r\cos(t)\hat{j}$, $\|r'(t)\| = r$, $T = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = -\sin(t)\hat{i} + \cos(t)\hat{j}$, and $\dfrac{dT}{dt} = -\cos(t)\hat{i} - \sin(t)\hat{j}$. Thus,

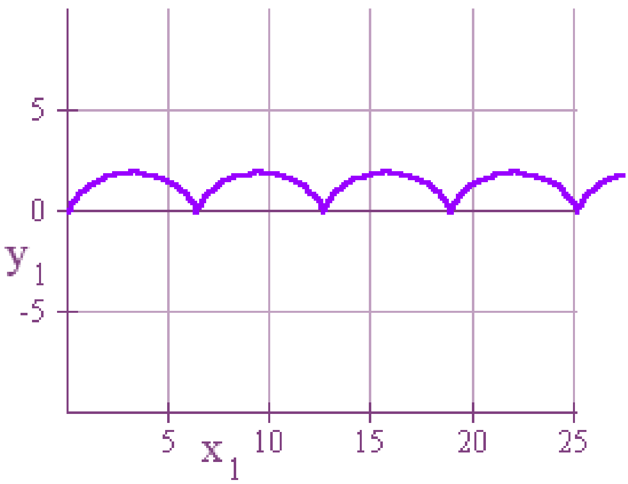
$$\kappa = \left\|\frac{dT}{ds}\right\| = \frac{\left\|\frac{dT}{dt}\right\|}{\left\|\frac{d\vec{r}}{dt}\right\|} = \frac{1}{r}$$

In other words, at any point on a circle of radius *r*, the curvature is $\frac{1}{r}$. This result helps us to better understand what curvature actually means. For example, if you think of the earth as a big circle with a large radius, then you can understand why it looks flat to you. With a radius of almost 4000 miles, the curvature is pretty darn small.

There exist several equivalent formulas that can be developed for calculating curvature, but this is as far as I'm going to go in this book. If you need an equivalent formula, it's always there in other books. For right now, though, I think the original formula is the best to focus on in order to better understand what curvature actually means.

Now let's go back and see how much of what we've learned can be applied to the formula that we derived for a cycloid generated by a circle of radius 1,

$$\vec{r}(t) = \left(t - \sin(t)\right)\hat{i} + \left(1 - \cos(t)\right)\hat{j}$$



Here are some important derivations,

$$\begin{aligned}
&\vec{r}(t) = \left(t - \sin(t)\right)\hat{i} + \left(1 - \cos(t)\right)\hat{j} \\
&\vec{r}'(t) = \left(1 - \cos(t)\right)\hat{i} + \sin(t)\,\hat{j} \\
&\left\|\vec{r}'(t)\right\| = \sqrt{1 - 2\cos(t) + \cos^2(t) + \sin^2(t)} = \sqrt{2 - 2\cos(t)}
\end{aligned}$$

From these equations it follows that,

$$\begin{aligned}
T(t) &= \frac{\vec{r}'(t)}{\left\|\vec{r}'(t)\right\|} = \frac{1 - \cos(t)}{\sqrt{2 - 2\cos(t)}}\hat{i} + \frac{\sin(t)}{\sqrt{2 - 2\cos(t)}}\hat{j} \\
N(t) &= \frac{\sin(t)}{\sqrt{2 - 2\cos(t)}}\hat{i} - \frac{1 - \cos(t)}{\sqrt{2 - 2\cos(t)}}\hat{j}
\end{aligned}$$

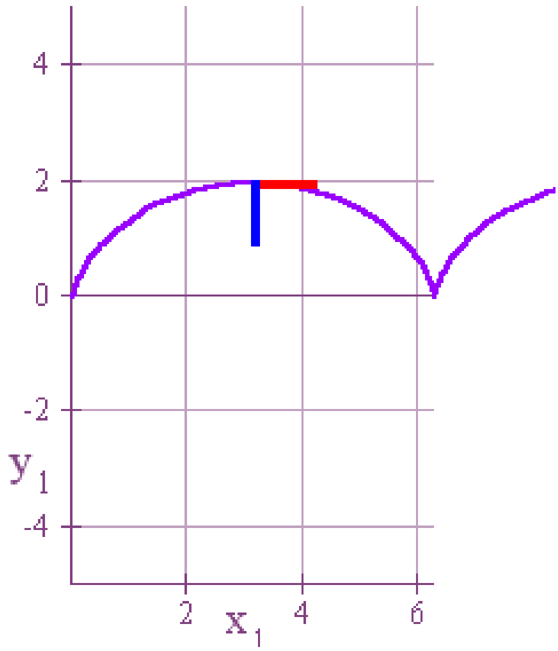
Hence, with a little (or a lot!) of effort we can obtain

$$T'(t) = \frac{1}{2}\frac{\sin(t)}{\sqrt{2-2\cos(t)}}\hat{i} - \frac{1}{2}\frac{1-\cos(t)}{\sqrt{2-2\cos(t)}}\hat{j}$$

$$\kappa(t) = \frac{\|T'(t)\|}{\|\vec{r}'(t)\|} = \frac{\sqrt{\frac{1}{4}\cdot\frac{\sin^2(t)}{2-2\cos(t)} + \frac{1}{4}\cdot\frac{1-2\cos t+\cos^2(t)}{2-2\cos(t)}}}{\sqrt{2-2\cos(t)}} = \frac{1}{2}\cdot\frac{1}{\sqrt{2-2\cos(t)}}$$

This means that when $t=\pi$, $\vec{r}(\pi)=\pi\hat{i}+2\hat{j}$, $T(\pi)=\hat{i}$, and $N(\pi)=-\hat{j}$. Let's graph these! The parametric equations we can use to graph $T(\pi)=2\hat{i}$ and $N(\pi)=-2\hat{j}$ so that they start at the point $(\pi,2)$ are,

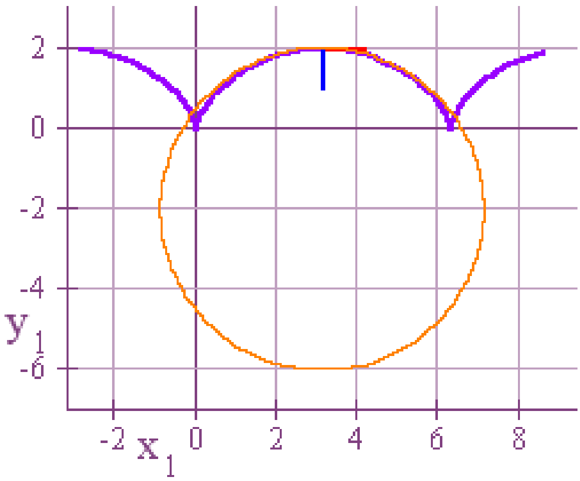
Unit Tangent	Unit Normal
$x=\pi+t$	$x=\pi$
$y=2$	$y=2-t$
$0\le t\le 1$	$0\le t\le 1$



Notice, as we mentioned before, that if you are facing in the direction of the unit tangent, then the unit normal is pointing to your right.

Now let's take a look at the curvature when $t = \pi$. We have that

$\kappa(\pi) = \frac{1}{2} \cdot \frac{1}{\sqrt{2 - 2\cos(\pi)}} = \frac{1}{4}$. That means that a circle of radius $\frac{1}{\kappa} = 4$ should fit into the curve at that point quite nicely. The center of such a circle should be at $(\pi, -2)$, four units below the point $(\pi, 2)$ when moving in the direction of the unit normal. Hence, we can parametrize this circle by finding a parametrization for a circle of radius 4 with center at the origin and then add the vector $\pi\hat{i} - 2\hat{j}$ to it. The end result is $\vec{w}(t) = (4\cos(t) + \pi)\hat{i} + (4\sin(t) - 2)\hat{j}$. Now just set *x* equal to the first component, *y* to the second component, and let $0 \le t \le 1$, and we've got our parametrization. Here's the final graph!



Close enough for government work!

Now let's see what we can accomplish with regard to arc length. To find the length of the arc from $t = 0$ to $t = 2\pi$, we're going to have to use some trigonometry identities, so let's get everything set up now. Recall that $\sin^2(t) = \dfrac{1-\cos(2t)}{2}$. This implies that $\sin^2\left(\dfrac{t}{2}\right) = \dfrac{1-\cos(t)}{2}$ which implies that $4\sin^2\left(\dfrac{t}{2}\right) = 2 - 2\cos(t)$. It's this last equation that we are going to have to use soon. We now have from our earlier work that the length of the arc from $t = 0$ to $t = 2\pi$ is

$$s(2\pi) = \int_0^{2\pi} \left\|\vec{r}'(t)\right\| dt = \int_0^{2\pi} \sqrt{2 - 2\cos(t)}\, dt = \int_0^{2\pi} \sqrt{4\sin^2\left(\frac{t}{2}\right)}\, dt = \int_0^{2\pi} 2\sin\left(\frac{t}{2}\right) dt$$

$$= -4\cos\left(\frac{t}{2}\right)\Big|_0^{2\pi} = 4 + 4 = 8$$

And it's just that simple!

While we're at it, let's also find the area under the curve from $t = 0$ to $t = 2\pi$. You would think this would be easy, but since we have our curve defined in terms of a parameter *t* instead of the usual $y = f(x)$, this will be a little tricky. First, let's recall our parametric equations for this cycloid,

$$x = t - \sin(t)$$
$$y = 1 - \cos(t)$$

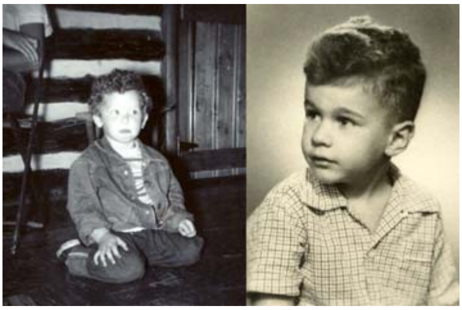
Observe, also, that if we did have *y* as a function of *x*, then the area under the curve would be given by $\int_{t=0}^{t=2\pi} y\, dx$. Now comes the clever part. If we compute $\frac{dx}{dt}$ above, then we get $\frac{dx}{dt} = 1 - \cos(t) = y$. Hence,

$$\int_{t=0}^{t=2\pi} y\, dx = \int_0^{2\pi} y \frac{dx}{dt} dt = \int_0^{2\pi} \left(1 - \cos(t)\right)\left(\left(1 - \cos(t)\right)\right) dt = \int_0^{2\pi} 1 - 2\cos(t) + \cos^2(t)\, dt$$

$$= \int_0^{2\pi} 1 - 2\cos(t) + \frac{1 + \cos(2t)}{2} dt = t - 2\sin(t) + \frac{t}{2} + \frac{\sin(2t)}{4}\Bigg|_0^{2\pi} = 3\pi$$

Now let's make an important definition and look at just one more thing. A smooth curve, geometrically speaking, is a curve without any sharp points on the graph. Notice that our cycloid is not smooth since we get a sharp point at intervals of 2π. We also say that a curve is smooth on an interval if it has a smooth parametrization. By this we mean that there exists a parametrization $\vec{r}(t)$ with the property that $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq \vec{0}$, the zero vector. It's not hard to see why we want $\vec{r}'(t)$ to be continuous. After all, if $\vec{r}'(t)$ changed direction in an abrupt manner, then that could result in a sharp point. But why do we want $\vec{r}'(t) \neq \vec{0}$. Well, first of all, notice that for the cycloid where we have a sharp point at $t = 2\pi$, $\vec{r}'(2\pi) = \left(1 - \cos(2\pi)\right)\hat{i} + \sin(2\pi)\hat{j} = \vec{0}$. Recall also that $\vec{r}'(t) = \vec{v}(t) = velocity$. If you are walking along, then it's possible to let your velocity vary continuously and still have a sharp point on your path. All you have to do is gradually let your speed decrease to zero and then start up in a totally different direction. Thus, if we really don't want any sharp points on our curve, then we also have to require that $\vec{v}(t) = \vec{r}'(t) \neq \vec{0}$. And that's the truth!

Christopher Benton is from parts unknown and grew up Deepinthehearta, Texas. After many adventures and misadventures, he finally became an adult and obtained a master of science degree and a doctor of philosophy degree in mathematics from the University of Houston. He currently lives in Arizona in The Valley of the Sun with his



beloved wife, Susan, and their dog, Dr. Chloe Continuum, Do.G. (doctor of growlology). Chloe's favorite pastimes are eating tasty dog food and curling up every evening with a good calculus book to read.