THE THIRD ISOMORPHISM THEOREM

Discussion: This Third Isomorphism Theorem is in some ways a continuation of our Correspondence Theorem in that it establishes an isomorphism between a quotient group and a particular quotient of another quotient group.

The Third Isomorphism Theorem: Let G be a group, let N and H be normal subgroups of G, and suppose that $N \subseteq H \subseteq G$. Then H/N is a normal subgroup of G/N, and $(G/N)/(H/N) \cong G/H$.

Proof: It follows immediately from the Correspondence Theorem that H/N is a normal subgroup of G/N. Now let $i : G \to G/N$ be the natural homomorphism, and let $j : G/N \to (G/N)/(H/N)$ be another natural homomorphism. Then $j \circ i$ is a homomorphism from G onto $(G/N)/(H/N)$.

$$G \xrightarrow{\ i\ } G/N \xrightarrow{\ j\ } (G/N)/(H/N)$$

Hence, our First Isomorphism Theorem tells us that $(G/N)/(H/N)$ is isomorphic to $G/Ker(j \circ i)$. Thus, we just need to figure out what is contained in $Ker(j \circ i)$. Thus, let $h \in H \subseteq G$. Then $Nh \in H/N \subseteq G/N$ tells us that $h \in Ker(j \circ i)$. On the other hand, if $g \in G$, but $g \notin H$, then $Ng \notin H/N$, and, thus, $g \notin Ker(j \circ i)$. Therefore, $Ker(j \circ i) = H$, and by the First Isomorphism Theorem, G/H is isomorphic to $(G/N)/(H/N)$.

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