THE SECOND ISOMORPHISM THEOREM

Discussion: In Theorem 29 we proved that if H is a subgroup of a group G and if N is a normal subgroup of G, then the right (left) cosets corresponding to elements of H form a subgroup of G/N. The theorem below, known as the Second Isomorphism Theorem, give us much sharper detail on the structure of this subgroup of G/N.

The Second Isomorphism Theorem: If H and N are subgroups of a group G with N normal in G, then $H/H \cap N \cong HN/N$.

Proof: Recall that earlier we proved that if H is a subgroup of G, then there will exist a corresponding subgroup of G/N that is obtained by looking at the cosets Nh where $h \in H$. This theorem, the Second Isomorphism Theorem, sharpens and clarifies the result. To prove it, though, we first need to show that $H \cap N$ is a normal subgroup of H and that HN is a subgroup of G that contains N. So let's begin!

To show that $H \cap N$ is a normal subgroup of H, we first need to show that it is at least a subgroup by verifying properties of closure and existence of inverses. Thus, let $n_1, n_2 \in H \cap N$. Since $n_1, n_2 \in H$, a subgroup of G, it follows that $n_1 n_2 \in H$. But by the same token, $n_1, n_2 \in N$ implies that $n_1 n_2 \in N$. Hence, $n_1 n_2 \in H \cap N$, and closure is satisfied.

Now suppose that $n \in H \cap N$. Then an inverse to n exists in both H and in N. In other words, $n^{-1} \in H$ and $n^{-1} \in N$ implies that $n^{-1} \in H \cap N$. Thus, existence of inverses is satisfied, and $H \cap N$ is a subgroup.

To show that $H \cap N$ is a normal subgroup of H, let $h \in H$ and let $n \in H \cap N$. Then $h^{-1}nh \in H$ since all three elements belong to H. But on the other hand, $h^{-1}nh \in N$ since N is a normal subgroup of G. Hence, $h^{-1}nh \in H \cap N$, and so $H \cap N$ is a normal subgroup of H.

Now let's show that HN is a subgroup of G. Thus, to show closure, let $h_1 n_1, h_2 n_2 \in HN$, and consider the product $h_1 n_1 h_2 n_2$. Since N is a normal subgroup of G, every left coset of N is equal to the corresponding right coset, and that means that $h_2 N = N h_2 = N n_1 h_2$. Hence, there exists $n_3 \in N$ such that $n_1 h_2 = h_2 n_3$. Thus, $h_1 n_1 h_2 n_2 = h_1 h_2 n_3 n_2 \in HN$, and closure is satisfied. To show the existence of inverses in HN, let $hn \in HN$. Then it's inverse is $n^{-1}h^{-1}$. However, again since N is normal in G, there exists $n_4 \in N$ such that $n^{-1}h^{-1} = h^{-1}n_4 \in HN$. Therefore, inverses exist in HN, and HN is a subgroup of G. Furthermore, $N \subseteq HN$ since every element of N can be written as $e \cdot n$ where $e \in H$ and $n \in N$.

And finally, we need to state and prove our isomorphism from $H/H \cap N$ to HN/N. In this case, define $f: H/H \cap N \to HN/N$ by $f[(H \cap N)h] = Nh$. To show that *f* is a homomorphism, observe that $f[(H \cap N)h_1] \cdot f[(H \cap N)h_2] = Nh_1 \cdot Nh_2 = N(h_1h_2) = f[(H \cap N)h_1h_2]$ Notice, too, that elements in $H/H \cap N$ look like $\{H \cap N, (H \cap N)h_1, (H \cap N)h_2, (H \cap N)h_3, \ldots\}$ where $h_1, h_2, h_3, \ldots \notin H \cap N$, and the corresponding elements in HN/N look like $\{N, Nh_1, Nh_2, Nh_3, \ldots\}$. From this it should be clear that $Ker(f) = H \cap N$ because if $h \notin H \cap N$, then it gets mapped to $Nh \neq N$, the identity in HN/N. Thus, from previous proof on homomorphisms and one-to-one functions, it follows that *f* is one-to-one. And finally, to show that *f* is onto, suppose that $Nhn \in HN/N$. Then since *N* is a normal subgroup, we can rewrite *hn* as n_1h for some $n_1 \in N$. Hence, $Nhn = Nn_1h = Nh = f[(H \cap N)h]$, and therefore, *f* is onto and $H/H \cap N \cong HN/N$.

□