EVEN AND ODD PERMUTATIONS – ANSWER

Theorem: If G acts on a set X, then every permutation in G is either even or odd, but not both.

Proof: To prove this, we are going to take for granted some facts about matrices from linear algebra, and that's both good and bad. It's bad in the sense that I am going to use some results without proving them, but it's good in that it shows how one branch of mathematics can be used to give a fairly simple proof in another branch. We'll start, though, with some background information and notation. Thus, let's consider S_3, the group of all permutations of three objects. We've previously used cycle notation like $(1,2,3)$ to indicate that whatever is in position 1 goes to position 2, whatever is in position 2 goes to position 3, and whatever is in position 3 goes to position 1. Another way to represent this, however, is by matrix multiplication. Also, if we label our objects as 1, 2, and 3 with 1 starting in position 1, 2 starting in position 2, and 3 starting in position 3, then we can express this starting position by the following column matrix.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{matrix} \leftarrow \text{ position 1} \\ \leftarrow \text{ position 2} \\ \leftarrow \text{ position 3} \end{matrix}$$

The permutation represented by the cycle $(1,2,3)$ can now be rewritten in terms of multiplication by a matrix. In particular,

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \begin{matrix} \leftarrow \text{ position 1} \\ \leftarrow \text{ position 2} \\ \leftarrow \text{ position 3} \end{matrix}$$

Notice that this matrix multiplication does exactly what we want. It moves whatever is in position 1 to position 2, whatever is in position 2 to position 3, and whatever is in position 3 to position 1. Furthermore, if we don't want to move anything at all, then we just multiply by the identity matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{matrix} \leftarrow \text{ position 1} \\ \leftarrow \text{ position 2} \\ \leftarrow \text{ position 3} \end{matrix}$$

When we are dealing with permutations of n objects, then every permutation can expressed in terms of multiplication by an $n \times n$ matrix where each row and column consists of exactly one entry set to 1 and the rest are set to 0. Such a matrix is appropriately called a permutation matrix.

Recall now that every square matrix has a number associated with it that is called its determinant. Recall also that the original application of determinants was to "determine" if a system of *n* equations in *n* unknowns had a unique solution or not. If the determinant is 0, then a solution either fails to exist or is not unique. On the other hand, if the determinant is not equal to zero, then the solution to the system is unique. However, this is only one application of determinants. Over the years, mathematicians have found many more.

For example, the determinant of a permutation matrix is always either 1 or -1. Thus,

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1,$$

but if we switch just two rows, then the determinant changes to -1. Hence,

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$$

Switching just two rows corresponds to a single transposition which, of course, is an odd permutation. In the example above, we switched rows 1 & 2, but if we follow that by switching rows 2 & 3, then the net result is two row switches and the resulting determinate returns back to 1. The permutation matrix also corresponds to the product of two transpositions since we switched two rows twice, and hence, the permutation is even.

$$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1$$

We can now give a very precise definition of what we mean by an even or odd permutation.

<u>Definition:</u> Let *A* be the permutation matrix corresponding to a permutation of *n* objects. Then our permutation is even if $|A| = 1$ and odd if $|A| = -1$.

And now to complete our proof, we just need one more fact from linear algebra.

<u>FACT:</u> Let *A* and *B* be $n \times n$ matrices. Then $|AB| = |A||B|$. In other words, the determinant of a product is equal to the product of the determinants.

And finally, recalling that every permutation can be written as a product of transpositions, let's suppose that we have a permutation of *n* objects that can be written both as the product of an even number of transpositions and as the product of an odd number of transpositions. To make this more concrete, let's suppose that our permutation is written in matrix form as a product of matrices such that each matrix corresponds to a single transposition, and let's suppose we can do this in two ways, as $A \cdot B \cdot C$ and as $D \cdot E$. If we can do this, then the permutation is both odd and even since it can be written as either the product of three transpositions or as the product of two transpositions. However, if $A \cdot B \cdot C = D \cdot E$, then $E^{-1} \cdot D^{-1} \cdot A \cdot B \cdot C = I$, then identity matrix. But we now have a contradiction since $1 = |I| = |E^{-1} \cdot D^{-1} \cdot A \cdot B \cdot C| = (-1)(-1)(-1)(-1)(-1) = -1$. Hence, our assumption that some permutation could be written as a product of either an even or odd number of transpositions is wrong, and, therefore, every permutation in *G* is either even or odd, but not both.

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