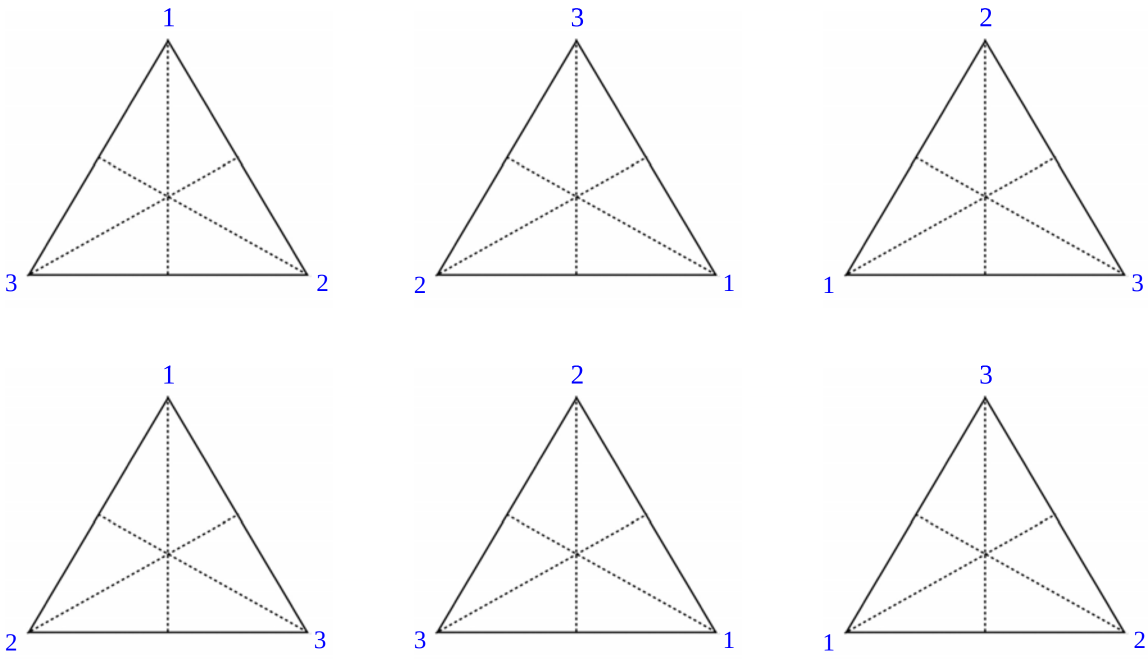
CAYLEY'S THEOREM – ANSWER

Theorem: Every finite group G is isomorphic to a group of permutations acting on a set of objects.

Proof: Instead of a more formal argument, we'll simply take a typical finite group and show how to find a permutation group that is isomorphic to it. In particular, let's look at D_3, the group of symmetries of an equilateral triangle.



This group is generated by rotations about the center flips about various axes of symmetry. Also, below is a multiplication table for D_3

	(1)(2)(3)	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
(1)(2)(3)	(1)(2)(3)	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
(1 2)	(1 2)	(1)(2)(3)	(1 2 3)	(1 3 2)	(1 3)	(2 3)
(1 3)	(1 3)	(1 3 2)	(1)(2)(3)	(1 2 3)	(2 3)	(1 2)
(2 3)	(2 3)	(1 2 3)	(1 3 2)	(1)(2)(3)	(1 2)	(1 3)
(1 2 3)	(1 2 3)	(2 3)	(1 2)	(1 3)	(1 3 2)	(1)(2)(3)
(1 3 2)	(1 3 2)	(1 3)	(2 3)	(1 2)	(1)(2)(3)	(1 2 3)

Furthermore, if we use letters to represent the various rotations and flips, then we can rewrite our multiplication table as follows.

$$\begin{aligned} e &= (1)(2)(3) \\ R &= (1\ \ 2\ \ 3) \\ R^2 &= (1\ \ 3\ \ 2) \\ F &= (2\ \ 3) \\ FR &= (1\ \ 2) \\ FR^2 &= (1\ \ 3) \end{aligned}$$

	e	R	R^2	F	FR	FR^2
e	e	R	R^2	F	FR	FR^2
R	R	R^2	e	FR^2	F	FR
R^2	R^2	e	R	FR	FR^2	F
F	F	FR	FR^2	e	R	R^2
FR	FR	FR^2	F	R^2	e	R
FR^2	FR^2	F	FR	R	R^2	e

When we look at this table, we notice that each row is a permutation of elements in the very first row. However, this does not mean that we are going to say that *R* is given by the following permutation:

$$R = \begin{pmatrix} e & R & R^2 & F & FR & FR^2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ R & R^2 & e & FR^2 & F & FR \end{pmatrix} = (e, R, R^2)(F, FR^2, FR)$$

No, instead we have to be a little more sophisticated so that things will work out easily in the end. In particular, remember that we want to thinks of our initial elements as occupying positions. Thus, *e* is in the first position, *R* is in the second position, R^2 is in the third position, *F* is in the fourth position, *FR* is in the fifth position, and FR^2 is in the sixth position.

	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}
	e	R	R^2	F	FR	FR^2
e	e	R	R^2	F	FR	FR^2
R	R	R^2	e	FR^2	F	FR
R^2	R^2	e	R	FR	FR^2	F
F	F	FR	FR^2	e	R	R^2
FR	FR	FR^2	F	R^2	e	R
FR^2	FR^2	F	FR	R	R^2	e

We can now set up our permutations correctly. In the maneuver R, he first element, e, moves from position one to position three which corresponds to R^2.

	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}
	e	R	R^2	F	FR	FR^2
e	e	R	R^2	F	FR	FR^2
R	R	R^2	e	FR^2	F	FR
R^2	R^2	e	R	FR	FR^2	F
F	F	FR	FR^2	e	R	R^2
FR	FR	FR^2	F	R^2	e	R
FR^2	FR^2	F	FR	R	R^2	e

The element in the third position, R^2, moves to the second position which corresponds to R.

	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}
	e	R	R^2	F	FR	FR^2
e	e	R	R^2	F	FR	FR^2
R	R	R^2	e	FR^2	F	FR
R^2	R^2	e	R	FR	FR^2	F
F	F	FR	FR^2	e	R	R^2
FR	FR	FR^2	F	R^2	e	R
FR^2	FR^2	F	FR	R	R^2	e

And the element in the second position, R, moves to the first position which corresponds to e.

	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}
	e	R	R^2	F	FR	FR^2
e	e	R	R^2	F	FR	FR^2
R	R	R^2	e	FR^2	F	FR
R^2	R^2	e	R	FR	FR^2	F
F	F	FR	FR^2	e	R	R^2
FR	FR	FR^2	F	R^2	e	R
FR^2	FR^2	F	FR	R	R^2	e

In other words, so far, we have (e, R^2, R). Continuing, we see that the element originally in the fourth position, F, moves to the fifth position which corresponds to FR.

	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}
	e	R	R^2	F	FR	FR^2
e	e	R	R^2	F	FR	FR^2
R	R	R^2	e	FR^2	F	FR
R^2	R^2	e	R	FR	FR^2	F
F	F	FR	FR^2	e	R	R^2
FR	FR	FR^2	F	R^2	e	R
FR^2	FR^2	F	FR	R	R^2	e

The element originally in the fifth position, FR, moves to the sixth position which corresponds to FR^2.

	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}
	e	R	R^2	F	FR	FR^2
e	e	R	R^2	F	FR	FR^2
R	R	R^2	e	FR^2	F	FR
R^2	R^2	e	R	FR	FR^2	F
F	F	FR	FR^2	e	R	R^2
FR	FR	FR^2	F	R^2	e	R
FR^2	FR^2	F	FR	R	R^2	e

And the element in the sixth position, FR^2 , moves to the fourth position which corresponds to *F*.

	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}
	e	R	R^2	F	FR	FR^2
e	e	R	R^2	F	FR	FR^2
R	R	R^2	e	FR^2	F	FR
R^2	R^2	e	R	FR	FR^2	F
F	F	FR	FR^2	e	R	R^2
FR	FR	FR^2	F	R^2	e	R
FR^2	FR^2	F	FR	R	R^2	e

Thus, the complete permutation for *R* is $R = (e, R^2, R)(F, FR, FR^2)$. Similarly, the permutation for *F*, when we construct it by thinking of the positions that our original elements get move to, is $F = (e, F)(R, FR)(R^2, FR^2)$. Now from our multiplication table we can see that $RF = FR^2$, and this latter element corresponds to the permutation $RF = FR^2 = (e, FR^2)(R, F)(R^2, FR)$. And finally, if we manually multiply our permutations, then we get $RF = (e, R^2, R)(F, FR, FR^2)(e, F)(R, FR)(R^2, FR^2) = (e, FR^2)(R, F)(R^2, FR) = FR^2$.

So what does this show us? Well, we've demonstrated how to convert each element in our group to a permutation that acts upon the elements of the group, and we've shown that a product such as $RF = FR^2$ gives us the same result, $RF = (e, R^2, R)(F, FR, FR^2)(e, F)(R, FR)(R^2, FR^2) = (e, FR^2)(R, F)(R^2, FR) = FR^2$, when we express our group elements as permutations. Therefore, every finite group *G* is isomorphic to a group of permutations acting on a set of group elements themselves.

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