ANOTHER CORRESPONDENCE THEOREM – ANSWER

Theorem: Let G be a group, N a normal subgroup of G, and let M be a subgroup of G/N that contains N. Also, define $f: G \to G/N$ by $f(g) = Ng$, and define $f^{-1}: G/N \to G$ by $f^{-1}(Ng) = \{g \in G \mid f(g) \in Ng\}$. Similarly, for any set $A \subseteq G/N$ let $f^{-1}(A) = \{g \in G \mid f(g) \in A\}$. Then if M is a subgroup of G/N, $f^{-1}(M)$ is a subgroup of G.

Proof: Let M be a subgroup of G/N and let f be defined as above. Then to show that $f^{-1}(M)$ is a subgroup of G we need to verify closure and existence of inverses. Thus, suppose that $h_1, h_2 \in f^{-1}(M)$. Then $f(h_1) = Nh_1$ and $f(h_2) = Nh_2$. Furthermore, since M is a subgroup of G/N, we have that $Nh_1 \cdot Nh_2 = N(h_1h_2) \in M$. But this means that $h_1h_2 \in f^{-1}(M)$, and so $f^{-1}(M)$ is closed under multiplication.

Now let $h \in f^{-1}(M)$. Then h has an inverse, h^{-1}, in the group G. Now consider $Nh \in M$. The element $Nh \in M$ has an inverse Nb in M such that $Nh \cdot Nb = N(hb) = N \subseteq M$. This implies that $hb = n$ for some $n \in N \subseteq M$. But this now implies that $h(bn^{-1}) = e \Rightarrow h^{-1} = bn^{-1}$. Furthermore, since $b, n^{-1} \in f^{-1}(M)$, we now have that $h^{-1} \in f^{-1}(M)$, and, hence, inverses exist for elements in $f^{-1}(M)$. Therefore, we can now say that $f^{-1}(M)$ is a subgroup of G.

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