PRODUCT OF NORMAL SUBGROUPS – ANSWER

Theorem: Let G be a group, let M and N be normal subgroups of G such that $MN = G$ and $M \cap N = e$ (the identity). Then if $m_1, m_2 \in M$ and $n_1, n_2 \in N$ such that $m_1 n_1 = m_2 n_2$, it follows that $m_1 = m_2$ and $n_1 = n_2$. In other words, each element in G can be represented in a unique way as a product of an element in M with an element in N.

Proof: Let G be a group, let M and N be normal subgroups of G such that $MN = G$ and $M \cap N = e$ (the identity), and suppose that $m_1, m_2 \in M$ and $n_1, n_2 \in N$ with $m_1 n_1 = m_2 n_2$. Then $m_1 n_1 = m_2 n_2 \Rightarrow m_2^{-1} m_1 n_1 = n_2 \Rightarrow m_2^{-1} m_1 = n_2 n_1^{-1}$. But $m_2^{-1} m_1 \in M$ and $n_2 n_1^{-1} \in N$. Hence, if $m_2^{-1} m_1 = n_2 n_1^{-1}$, then $m_2^{-1} m_1, n_2 n_1^{-1} \in M \cap N$. But since $M \cap N = e$, it follows that $m_2^{-1} m_1 = e$ and $n_2 n_1^{-1} = e$. From this it follows that $m_1 = m_2$ and $n_2 = n_1$. Therefore, each element in G can be represented in a unique way as a product of an element in M with an element in N.

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