THE CENTER IS NORMAL – ANSWER

Definition: The center of a group *G*, denoted by $Z(G)$, is the set of all elements in *G* that commute with all other elements in *G*.

Theorem: The center of a group *G* is a normal subgroup of *G*.

Proof: First, note that the center of a group always exists since the identity element always belongs to the center (since it commutes with every other element in *G*). Second, we need to show that the center is a subgroup by showing that it is closed under multiplication and every element in the center has an inverse in the center. Thus, let $a,b \in Z(G)$ and let $c \in G$. Then $(ab)c = a(bc) = a(cb) = (ac)b = (ca)b = c(ab)$. Hence, since *ab* commutes with an arbitrary element of *G*, *ab* is in the center of *G*, and, thus, $Z(G)$ is closed under multiplication. Now let $a \in Z(G)$ and let $c \in G$. Then

$ac = ca \Rightarrow (ac)a^{-1} = (ca)a^{-1} \Rightarrow aca^{-1} = c(aa^{-1}) \Rightarrow aca^{-1} = c \Rightarrow a^{-1}(aca^{-1}) = a^{-1}c$

$\Rightarrow (a^{-1}a)ca^{-1} = a^{-1}c \Rightarrow eca^{-1} = a^{-1}c \Rightarrow ca^{-1} = a^{-1}c.$

Therefore, if *a* commutes with *c*, then a^{-1} commutes with *c*, and, thus, $a^{-1} \in Z(G)$ and $Z(G)$ is a subgroup of *G*.

To show that $Z(G)$ is a normal subgroup, let $a \in Z(G)$ and let $c \in G$. Then to show that $Z(G)$ is a normal subgroup of *G*, it suffices to show that $c^{-1}ac \in Z(G)$. But this is easy since, *a* commutes with every element in *G*. In other words, $c^{-1}ac = (c^{-1}a)c = (ac^{-1})c = a(c^{-1}c) = ae = a \in Z(G)$. Therefore, the center of a group *G* is a normal subgroup of *G*.

□