WHEN MULTIPLICATION IS NOT WELL DEFINED – ANSWER

<u>Theorem:</u> If H is a subgroup that is not a normal subgroup of G and $Ha_1 = Ha_2$ and $Hb_1 = Hb_2$, then Ha_1b_1 is not necessarily equal to Ha_2b_2.

<u>Proof:</u> If H is a subgroup of G, but H is not normal in G, then there exists at least one $a_1 \in G$ such that $= Ha_1 \neq a_1H$. In particular, that means that there are no $h_3, h_4 \in H$ such that $a_1h_4 = h_3a_1$. Now suppose that $Ha_1 = Ha_2$ and $Hb_1 = Hb_2$, and assume that $Ha_1b_1 = Ha_2b_2$. Then there exists $h_1, h_2 \in H$ such that $a_2 = h_1a_1$ and $b_2 = h_2b_1$. Hence, $Ha_1b_1 = Ha_2b_2 = H(h_1a_1)(h_2b_1) = (Hh_1)a_1h_2b_1 = Ha_1h_2b_1$. But this implies that there exists $h_3 \in H$ such that $a_1b_1 = h_3a_1h_2b_1$ which implies that $a_1 = h_3a_1h_2$. Now let $h_4 = h_2^{-1} \in H$. Then $a_1 = h_3a_1h_2 \Rightarrow a_1h_2^{-1} = h_3a_1 \Rightarrow a_1h_4 = h_3a_1$. But this contradicts our initial assumption about a_1. Therefore, if H is a subgroup that is not a normal subgroup of G and $Ha_1 = Ha_2$ and $Hb_1 = Hb_2$, then Ha_1b_1 is not necessarily equal to Ha_2b_2 and, thus, our multiplication of cosets is not well-defined.

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