WHEN MULTIPLICATION IS WELL DEFINED – ANSWER

Theorem: If H is a normal subgroup of G and $Ha_1 = Ha_2$ and $Hb_1 = Hb_2$, then $Ha_1b_1 = Ha_2b_2$.

Proof: Suppose hat H is a normal subgroup of G. Then for every $a \in G$, we have that $Ha = aH$. That means that for every product ha where $h \in H$, there exists $h_1 \in H$ such that $ah_1 = ha$. Now suppose that $Ha_1 = Ha_2$. Then $a_1, a_2 \in Ha_1 = Ha_2$, and, hence, there exists $h_2 \in H$ such that $a_1 = h_2a_2$. In a similar manner, if $b_1, b_2 \in Hb_1 = Hb_2$, then there exists $h_3 \in H$ such that $b_1 = h_3b_2$. Putting this all together, we can now conclude that $Ha_1b_1 = H(h_2a_2)(h_3b_2) = Hh_2(a_2h_3)b_2 = Hh_2(h_4a_2)b_2 = Hh_2h_4(a_2b_2) = Ha_2b_2$. What this means is that when H is a normal subgroup, we can define multiplication of cosets in a way that is independent of which representative of that coset we pick. And this is what we mean when we say that the multiplication is well-defined.

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