COSETS AND EQUIVALENCE RELATIONS – ANSWER

Definition: If X is a non-empty set, then a relation between elements in X is called an equivalence relation if and only if the following conditions are met:

1. For every $a \in X$, $a \equiv a$ (reflexive),
2. For every $a, b \in X$, if $a \equiv b$, then $b \equiv a$ (symmetric), and
3. For every $a, b, c \in X$, if $a \equiv b$ and $b \equiv c$, then $a \equiv c$ (transitive).

Theorem: If H is a subgroup of a group G, then the right (left) cosets of H in G define an equivalence relation.

Proof: The easy way to prove this is to simply note that the intersection of any two right (left) cosets is the null set and the union of the right (left) cosets gives us back all of G. Hence, the cosets form a partition of G into disjoint intervals whose union is G, and, therefore, coset membership defines an equivalence relation. More specifically, previous proofs have shown that two right (left) cosets either have an empty intersection or they are equal to one another. Thus, it follows that (1) $Ha = Ha$, (2) if $Ha = Hb$, then $Hb = Ha$, and (3) if $Ha = Hb$ and $Hb = Hc$, then $Ha = Hc$. Hence, the right (left) cosets define an equivalence relation.

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