INTERSECTION OF COSETS – ANSWER

Theorem: If H is a subgroup of a finite group G, then any two right (left) cosets either coincide or have an empty intersection.

Proof: We will prove the theorem just for right cosets since the argument for left cosets is the same. Thus, let H is a subgroup of a finite group G and suppose that $a, b \in G$ and that Ha and Hb are right cosets. Recall that if H has m elements, $e = h_1, h_2, h_3, \ldots, h_m$,then the members of Ha are $a, h_2 a, h_3 a, \ldots, h_m a$ and the members of Hb are $b, h_2 b, h_3 b, \ldots, h_m b$. If $Ha \cap Hb = \varnothing$, then we're done. Thus assume that the intersection is non-empty. Then that means there exist $h_j a \in Ha$ and $h_k b \in Hb$ such that $h_j a = h_k b$. But this means that $a = h_j^{-1} h_k b$ and $b = h_k^{-1} h_j a$. Hence, every element in Hb can be written as a product of an element in H with a, and every element in Ha can be written as a product of an element in H with b. From this it follows that every element in Hb is also an element in Ha, and every element in Ha is also an element in Hb. Thus, $Ha = Hb$, and, in general, for any two right cosets Ha and Hb, either $Ha \cap Hb = \varnothing$ or $Ha = Hb$.

□

NOTE: This proof can be extended to include infinite groups, but we don't want to get into the complexities of infinite sets at this point.