# THE KERNEL OF A HOMOMORPHISM

Discussion: Now that we've defined a homomorphism, we define the Kernel of our homomorphism to be the set of all elements in the first group that get sent to the identity element in the second group. Below we prove that this set, the Kernel, is not only a subgroup of our original group, it's also a normal subgroup, and that fact has major implications when it comes to investigating what homomorphisms from one group to another are even possible.

Definition: Let $f : A \to B$ be a homomorphism from a group *A* onto *B*. Then the Kernel of *f*, denoted by $Ker(f)$, is defined by
$Ker(f) = \{x \in A \mid f(x) = e \text{ where } e \text{ is the identity element in } B\}$.

Theorem: Let $f : A \to B$ be a homomorphism from a group *A* onto *B*. Then $Ker(f)$ is a normal subgroup of *A*.

Proof: First we will show that $Ker(f)$ is a subgroup of *A* by showing that it is closed under multiplication and that it contains inverses. Thus, let $a, b \in Ker(f)$. Then $e = f(a)f(b) = f(ab)$ implies that $ab \in Ker(f)$ and $Ker(f)$ is closed under multiplication. Now let $a \in Ker(f)$. Then there exists $a^{-1} \in A$. However, $e = f(e) = f(aa^{-1}) = f(a)f(a^{-1}) = e \cdot f(a^{-1}) = f(a^{-1})$ implies that $a^{-1} \in Ker(f)$, and, hence, $Ker(f)$ is a subgroup of *A*.

To show that $Ker(f)$ is a normal subgroup of *A*, let $g \in A$ and let $x \in Ker(f)$. Then $f(g^{-1}xg) = f(g^{-1})f(x)f(g) = f(g)^{-1} \cdot e \cdot f(g) = f(g)^{-1} f(g) = e$ implies that $g^{-1}xg \in Ker(f)$. Therefore, $Ker(f)$ is a normal subgroup of *A*.

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