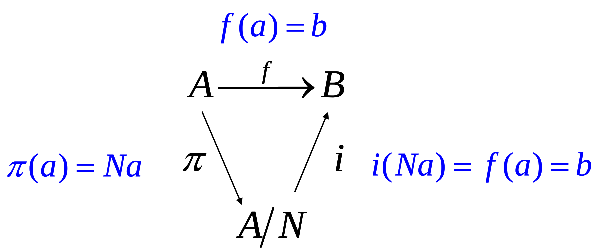
THE FIRST ISOMORPHISM THEOREM

Discussion: This first isomorphism theorem shows that every homomorphism from a group A onto a group B is essentially defined by a corresponding quotient group. Thus, if we know all the normal subgroups of a given group, then we essentially know all the homomorphisms that can be constructed from that group. This is a very powerful result!

The First Isomorphism Theorem: Let $f: A \to B$ be a homomorphism from a group A onto a group B, and let $N = Ker(f)$. Then $A/Ker(f) = A/N \cong B$.

Proof: Let $f: A \to B$ be a homomorphism from a group A onto a group B, and let $N = Ker(f)$. Recall that $\pi: A \to A/N$ defined by $\pi(a) = Na$ is called the natural homomorphism. Now define a function i from A/N to B by $i(Na) = f(a)$. For clarity, below is a diagram that illustrates the relationships between all of these functions.



$N = Ker(f)$

$$A/Ker(f) = A/N \cong f(A) = B$$

What we want to do now is to verify that $i: A/N \to B$ is an isomorphism. To do this, we need to show that i is a homomorphism, that i is onto, and that i is one-to-one. First, we will show that this function is a homomorphism. Thus, let $Nx, Ny \in A/N$. Then $i(NxNy) = i(Nxy) = f(xy) = f(x)f(y) = i(Nx)i(Ny)$. Hence, $i: A/N \to B$ is a homomorphism. Also, notice that it doesn't matter what representative element we choose to use from a coset such as Na. For instance, since $N = Ker(f)$, if $a, b \in Na$, then $a = nb$ and $f(a) = f(nb) = f(n)f(b) = e \cdot f(b) = f(b)$. Hence, it is also true that $i(Na) = i(Nb)$.

To show that $i: A/N \to B$ is onto, let $b \in B$. Then since $f: A \to B$ is onto, there exists $a \in A$ such that $f(a) = b$. Consequently, $i(\pi(a)) = i(Na) = f(a) = b$, and this shows that i is also onto.

To show that *i* is one-to-one, we will use our previous result that a homomorphism is one-to-one if and only if the Kernel of $i : A/N \to B$ consists of only the identity. Thus, suppose $Na \in Ker(i)$. Then one the one hand, $i(Na) = e$. But one the other hand, $i(Na) = f(a)$. Hence, $f(a) = e$ implies that $a \in Ker(f) = N$, and, thus, $Na = N$, the identity in A/N. Therefore, the Kernel of $i : A/N \to B$ consists of only the identity which implies that $i : A/N \to B$ is one-to-one, and, hence, $i : A/N \to B$ is an isomorphism.

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