THE CORRESPONDENCE THEOREM

Discussion: This important theorem basically delineates a lot of correspondences that exist between subgroups in a group G that contain a given normal subgroup N and subgroups in the corresponding quotient group G/N. In particular, this means that we can learn things about the structure of G by studying G/N.

The Correspondence Theorem: Let G be a group, let N be a normal subgroup of G, and let $\pi : G \to G/N$ be the natural homomorphism. Then,

1. If H is a subgroup of G such that $N \subseteq H$, then H/N is a subgroup of G/N.
2. If M is a subgroup of G/N, then $H = \pi^{-1}(M) = \{g \in G \mid \pi(g) \in M\}$ is a subgroup of G that contains N and $H/N = M$.
3. If H is a normal subgroup of G such that $N \subseteq H$, then H/N is a normal subgroup of G/N.
4. If H/N is a normal subgroup of G/N, then $H = \pi^{-1}(H/N) = \{g \in G \mid \pi(g) \in H/N\}$ is a normal subgroup of G.
5. If H and K are subgroups of G such that $N \subseteq H \subseteq K$, then $H/N \subseteq K/N$.
6. If $H/N \subseteq K/N$, then $N \subseteq H \subseteq K$.
7. If H and K are subgroups of G such that $N \subseteq H \subseteq K$, then $[K:H] = [K/N : H/N]$.
8. If $N \subseteq H \subseteq K$, where H and K are subgroups of G, and if H is normal in K, then H/N is normal in K/N.
9. If $N \subseteq H \subseteq K$, where H and K are subgroups of G, and if H/N is normal in K/N, then H is normal in K.

Proof: (1) Let H be a subgroup of G such that $N \subseteq H$ where N is a normal subgroup of G. To show that H/N is a subgroup of G/N, we just need to show closure and existence of inverses. Thus, suppose $Na, Nb \in H/N$. Then $a, b \in H$. Since H is a subgroup of G, there exists $c \in H$ such that $c = ab$. Hence, $NaNb = Nab = Nc \in H/N$, and, thus, closure is satisfied.

Now suppose $Na \in H/N$. Then $a \in H$, and since H is a subgroup of G, there exists $a^{-1} \in H$ such that $aa^{-1} = e$. Consequently, $Na^{-1} \in H/N$ and $NaNa^{-1} = Naa^{-1} = Ne = N$, the identity in H/N. Therefore, inverses also exist in H/N and H/N is a subgroup of G/N.

(2) Suppose M is a subgroup of G/N and let $H = \pi^{-1}(M) = \{g \in G \mid \pi(g) \in M\}$. Then clearly $N \subseteq H = \pi^{-1}(M) = \{g \in G \mid \pi(g) \in M\}$ since N is just π^{-1} applied to the identity element in M. To show that H is a subgroup of G, we need to verify closure and existence of inverses. Thus, suppose that $a, b \in H$. Then $Na, Nb \in M$ and

$NaNb = Nab \in M$. From this it follows that $ab \in H = \pi^{-1}(M) = \{g \in G \mid \pi(g) \in M\}$, and *H* is closed under multiplication.

To show that inverses exist in *H*, suppose $a \in H$. Then $Na \in M$ and because inverses exist in *M*, there exists $Nb \in M$ such that $NaNb = Nab = N$. However, this means both that $b \in H$ and $ab = n$ for some $n \in N$. But this also implies that $(ab)n^{-1} = a(bn^{-1}) = e$, the identity element in *G*, and, thus, $bn^{-1} = a^{-1}$. We can now conclude that since $b \in H$ and $n, n^{-1} \in N \subseteq H$ that $bn^{-1} = a^{-1} \in H$, and, therefore, *H* is a subgroup of *G* that contains *N*.

Finally, since $H = \pi^{-1}(M) = \{g \in G \mid \pi(g) \in M\}$ and since $N = Ker(\pi)$ it follows immediately that $\pi(H) = M = H/N$,

(3) Suppose that *H* is a normal subgroup of *G* such that $N \subseteq H$, and consider H/N , a subgroup of G/N . If $Ng \in G/N$ and $a \in H$, then $Ng^{-1}NaNg = Ng^{-1}ag \in H/N$ since *H* being normal in *G* tells us that $g^{-1}ag \in H$ and, hence, $N(g^{-1}ag) \in H/N$. Therefore, H/N is a normal subgroup of G/N .

(4) Suppose H/N is a normal subgroup of G/N where $H = \pi^{-1}(H/N) = \{g \in G \mid \pi(g) \in H/N\}$, and let $Ng \in G/N$ where $g \in G$. By part (2) above, we know that *H* is a subgroup of *G*. Now, if $Na \in H/N$, then $Ng^{-1}NaNg = Ng^{-1}ag \in H/N$ since H/N is a normal subgroup of G/N . But this means that $g^{-1}ag \in \pi^{-1}(Ng^{-1}ag) \subseteq H$. Therefore, *H* is a normal subgroup of *G*.

(5) Suppose *H* and *K* are subgroups of *G* such that $N \subseteq H \subseteq K$. Then H/N and K/N are both subgroups of G/N . Furthermore, if $N \subseteq H \subseteq K$, then $a \in H$ implies that $a \in K$, and this in turn means that if $Na \in H/N$, then $Na \in K/N$. Therefore, $H/N \subseteq K/N$.

(6) Suppose $H/N \subseteq K/N$. Then $H = \pi^{-1}(H/N) = \{g \in G \mid \pi(g) \in H/N\}$ and $K = \pi^{-1}(K/N) = \{g \in G \mid \pi(g) \in K/N\}$ are subgroups of *G*. Since $N \in H/N \subseteq K/N$, it follows immediately that
$N \subseteq H = \pi^{-1}(H/N) = \{g \in G \mid \pi(g) \in H/N\} \subseteq \pi^{-1}(K/N) = \{g \in G \mid \pi(g) \in K/N\} = K$.

(7) Suppose *H* and *K* are subgroups of *G* such that $N \subseteq H \subseteq K$. Then *H* is also a subgroup of *K*, and H/N and K/N are both subgroups of G/N with $H/N \subseteq K/N$. Consequently, H/N is also a subgroup of K/N . Furthermore, by Lagrange's Theorem, $[K:H] = \frac{|K|}{|H|}$ and $[K/N : H/N] = \frac{|K/N|}{|H/N|} = \frac{|K|/|N|}{|H|/|N|} = \frac{|K|}{|H|}$. Therefore, $[K:H] = [K/N : H/N]$.

(8) Suppose that *H* is normal in *K* where $N \subseteq H \subseteq K$. If $Na \in H/N$ and $Ng \in K/N$, then $g^{-1}ag \in H$ since *H* is normal in *K*. Thus, $Ng^{-1}NaNg = Ng^{-1}ag \in H/N$ and, therefore,

H/N is normal in K/N.

(9) Suppose H/N is normal in K/N where $N \subseteq H \subseteq K$ are all subgroups of *G*. If we simply restrict ourselves to the subgroup *K*, then it immediately follows from (4) above that *H* is normal in *K*.

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