SYMBOLIC LOGIC – ANSWERS

This lesson is an introduction to symbolic logic and what we actually mean in mathematics by statements such as "*a* implies *b*" and "*a* if and only if *b*." Below are some common symbols that are used in logic followed by the corresponding math symbols that I will use instead.

LOGIC	**MATH**
~ or $\neg$	not
$\vee$	or
$\wedge$	and
$a \to b$	$a \Rightarrow b$
$a \leftrightarrow b$	$a \Leftrightarrow b$

The statement "$a \Rightarrow b$" can be read as *"a implies b"* or *"if a then b"* or *"a is a sufficient condition for b"* or *"b is a necessary condition for a."*

The statement "$a \Leftrightarrow b$" can be read or written as *"a iff b"* or *"a if and only if b"* or *"a is a necessary and sufficient condition for b."*

Using the logical connectives above, we can rewrite "$a \Rightarrow b$" as "not (a & not-b)." Similarly, since "$a \Leftrightarrow b$" means "$a \Rightarrow b$ & $b \Rightarrow a$," we can rewrite "$a \Leftrightarrow b$" as "[not (a & not-b)] & [not (b & not-a)]."

In mathematics, for the compound statement *"A & B"* to be true, both of the statements *A* and *B* must be true. On the other hand, for the compound statement *"A or B"* to be true, only one of the statements must be true. Construct some simple examples to convince yourself that this is the correct way to proceed. Also, in mathematics, unless stated otherwise, we always use an *inclusive or*. That means that for *"A or B"* to be true, we either have *A* true or *B* true or both *A* and *B* true. In an *exclusive or*, either *A* or *B* can be true, but not both at the same time.

Now, here are some things for you to either do or look up and respond to.

1. What is a *truth table?*

   A truth table is a 2-dimensional array that indicates the final truth value (true or false) of a statement based upon the truth values assigned to its component propositions.

   See http://mathworld.wolfram.com/TruthTable.html

2. What is the *law of the excluded middle?*

   The Law of the Excluded Middle assumes that for any proposition P either it or its negation is true and that there are no other possibilities. In other words, A or not-A. A proposition is assumed to be either true or false.

   See http://mathworld.wolfram.com/LawoftheExcludedMiddle.html

3. What is a *tautology?*

   A tautology is a statement that is always true regardless of whether its component propositions are true or false. Hence, in a truth table, the final truth value will always be "true." For example, consider the statement, "I will study math or I won't study math."

   See http://mathworld.wolfram.com/Tautology.html

4. What is a *contradiction?*

   A contradiction is a statement of the form (P & not-P). If we have an argument that implies both P and not-P, then the argument has led to a contradiction. Alternatively, in a truth table, the final truth value will always be "false."

   See http://mathworld.wolfram.com/Contradiction.html

5. Complete the following *truth tables*. Your final values should be in the column shaded yellow.

Not	**A**
F	T
T	F

A	&	B
T	T	T
T	F	F
F	F	T
F	F	F

A	or	B
T	T	T
T	T	F
F	T	T
F	F	F

Not	[A	&	(Not	B)]
T	T	F	F	T
F	T	T	T	F
T	F	F	F	T
T	F	F	T	F

{ Not	[A	&	(Not	B)] }	&	{ Not	[B	&	(Not	A)] }
T	T	F	F	T	T	T	T	F	F	T
F	T	T	T	F	F	T	F	F	F	T
T	F	F	F	T	F	F	T	T	T	F
T	F	F	T	F	T	T	F	F	T	F

6. What is *modus ponens?* Give an example.

   Modus ponens is a form of argument that essential says that if *F implies G* (i.e. $F \Rightarrow G$) and if *F* is true, then *G* is true. The steps are,
   (1) $F \Rightarrow G$
   (2) F
   (3) Therefore, *G*.

   Example:
   If someone is a man, then they are mortal.
   Socrates is a man.
   Therefore, Socrates is mortal.

7. What is *modus tollens*" Give an example.

   Modus tollens is a form of argument that essential says that if *F implies G* (i.e. $F \Rightarrow G$) and if *G* is false, then *F* is false. The steps are,
   (1) $F \Rightarrow G$
   (2) not *G*
   (3) Therefore, not *F*.

   Example:
   If I am at home, then I will study math.
   I am not studying math.
   Therefore, I am not at home.

8. The statement $A \Rightarrow B$ means "$\text{not } (A \,\&\, \text{not } B)$." Use a truth table to explain why a false statement can imply anything? In other words, why is a statement such as "*If the moon is made of green cheese, then I am the smartest man on the planet*" considered to be true?

   Below is the truth table for $A \Rightarrow B$ (i.e. $\text{not } (A \,\&\, \text{not } B)$). From this table we can see that if *A* is false, then the final truth value of the statement is always "true." The implication is false only if *A* is true while *B* is false. Hence, a false statement can imply anything.

Not	[A	&	(Not	B)]
T	T	F	F	T
F	T	T	T	F
T	F	F	F	T
T	F	F	T	F

9. Use truth tables to show that " not (*a* & not-*b*) " and "not-*a* or b" are *logically equivalent*. In this context, logical equivalence means that if *a* and *b* are assigned the same truth values, then the truth values of our final compound statements are the same.

   We have previously completed the truth table for " not (*a* & not-*b*) ."

Not	[A	&	(Not	B)]
T	T	F	F	T
F	T	T	T	F
T	F	F	F	T
T	F	F	T	F

We just need to complete the corresponding truth table for "not-*a* or b."

(Not	A)	or	B
F	T	T	T
F	T	F	F
T	F	T	T
T	F	T	F

From the results we can see that the same true/false assignments made to *A* and *B* result in the same final truth value for both "$\text{not }(a \text{ \& not-}b)$" and "not-*a* or b." Therefore, there are logically equivalent.

10. Consider the statement *"This sentence is false."* What are the implications of this statement being true? What are the implications of it being false? Read http://en.wikipedia.org/wiki/Paradox.

    In general, we often think of a paradox as something that is simply unusual or unexpected, but in particular, we tend to think of a mathematical paradox as a statement that implies its negation such as $A \Leftrightarrow \text{not-}A$. This is the type of paradox we have above. If the sentence is true, then by definition, it is false, and if the sentence is false, then it must be true. Paradox! In many respects, I think of paradoxes as important because they reveal to us what may be either flaws or unexpected consequences of our logic. Sometimes they also seem to reveal the limitations of logical discourse. Furthermore, their presence suggests that sometimes something will exist other than just "true" or "false."

11. What is the *inverse* of $a \Rightarrow b$?
    The *inverse of* $a \Rightarrow b$ is $\text{not-}a \Rightarrow \text{not-}b$.

    What is the *converse* of $a \Rightarrow b$?
    The *converse of* $a \Rightarrow b$ is $b \Rightarrow a$.

    What is the *contrapositive* of $a \Rightarrow b$?
    The *contrapositive of* $a \Rightarrow b$ is $\text{not-}b \Rightarrow \text{not-}a$.

    Use truth tables to show that $a \Rightarrow b$ is *logically equivalent* to $\text{not-}b \Rightarrow \text{not-}a$.

    As stated previously, $a \Rightarrow b$ means "$\text{not }(a \text{ \& not-}b)$," and the truth table for this statement is below.

Not	**A**	**&**	**(Not**	**B)]**
T	T	F	F	T
F	T	T	T	F
T	F	F	F	T
T	F	F	T	F

Similarly, $\text{not-}b \Rightarrow \text{not-}a$ means "$\text{not }(\text{not-}b \text{ \& not-not-}a)$" which we can simplify to "$\text{not }(\text{not-}b \text{ \& } a)$." The truth table for this statement is below

Not	[(not	B)	&	A]
T	F	T	F	T
F	T	F	T	T
T	F	T	F	F
T	T	F	F	F

From the above tables we can see that when *A* and *B* are given the same truth values in each table, then the truth values of the resulting statements are identical. Therefore, $a \Rightarrow b$ is *logically equivalent* to $\text{not-}b \Rightarrow \text{not-}a$.

12. The truth table for "$\text{not } (a \,\&\, b)$" is:

Not	(A	&	B)
F	T	T	T
T	T	F	F
T	F	F	T
T	F	F	F

Similarly, the truth table for "not-*a* or not-*b*" is:

(Not	A)	or	(Not	B)
F	T	F	F	T
F	T	T	T	F
T	F	T	F	T
T	F	T	T	F

Therefore, $a \uparrow b$, i.e. "$\text{not } (a \,\&\, b)$," is logically equivalent to "not-*a* or not-*b*."