Lesson 9

SUBGROUP LATTICES

Another way to visually present information about a group is through the lattice of subgroups. This is a diagram which shows all the subgroups a given group contains, and it's arranged with the group at the top and the identity at the bottom. Also, lines are drawn to show which subgroups are contained in which. Additionally, when I draw a subgroup lattice, I like to indicate in cycle notation the elements in each subgroup. That way, you can not only see the number of elements in each subgroup, you can also sometimes determine that two subgroups with the same number of elements are not isomorphic if their elements have different orders.

The first few subgroup lattices we'll present are very simple due to the lack of many subgroups. For instance, the identity element forms a group of order 1, and you see only one set in the lattice, the set containing the identity.

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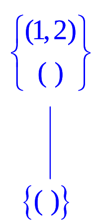
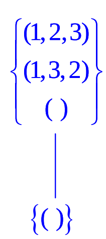
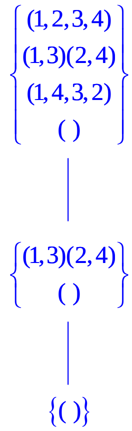
The cyclic group of order 2, C_2, is not much better. There are only two elements in this group, and the only subgroups are the entire group and the identity.



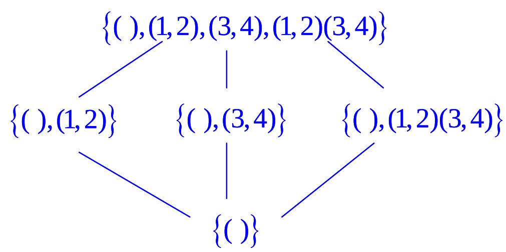
Similarly, the cyclic group of order 3, C_3, has only two subgroups.



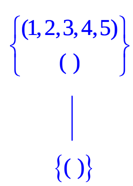
Things start to get a little more interesting when we get to groups of order 4, however, because now we have two possible groups, C_4 and the Klein 4-group. We'll first show the subgroup lattice for C_4, the cyclic group of order 4.



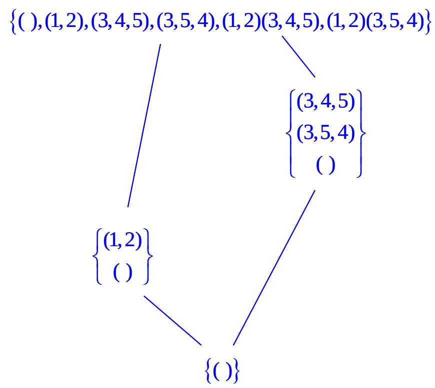
Next, we have the lattice for the Klein 4-group which we can think of as $C_2 \times C_2$. Notice that since the subgroup lattice for the Klein 4-group is vastly different from the subgroup lattice for C_4, the two groups cannot be isomorphic. They have totally different structures.



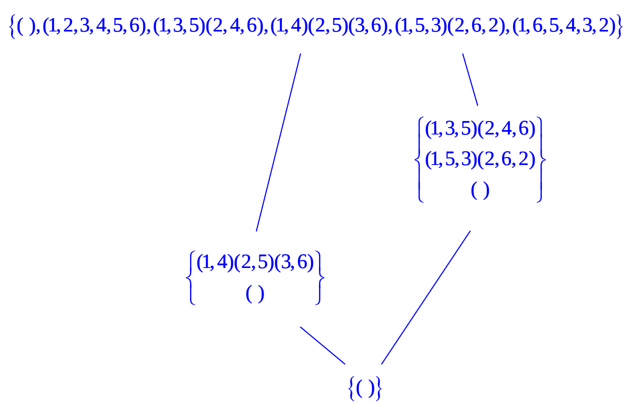
For C_5, the cyclic group of order 5, the subgroup lattice is again very simple. Recall that since the order of a subgroup must divide into the order of the group, it follows that C_5 has only two subgroups as a result of 5 being prime. The two subgroups are the entire group and the identity.



There are two groups of order 6, C_6 and $D_3 \cong S_3$. Recall, too, that $C_6 \cong C_2 \times C_3$. Thus, we can think of $(1,2)$ and $(3,4,5)$ as generators for $C_6 \cong C_2 \times C_3$. Using these generators, we get the following diagram.



On the other hand, if we use $(1,2,3,4,5,6)$ as the generator, then we obtain the following diagram.



Notice that even though we can represent $C_6 \cong C_2 \times C_3$ in two different ways via permutations, the subgroup lattices essentially look the same. And finally, here is the subgroup lattice for C_7 which, again, is very simple since 7 is prime.