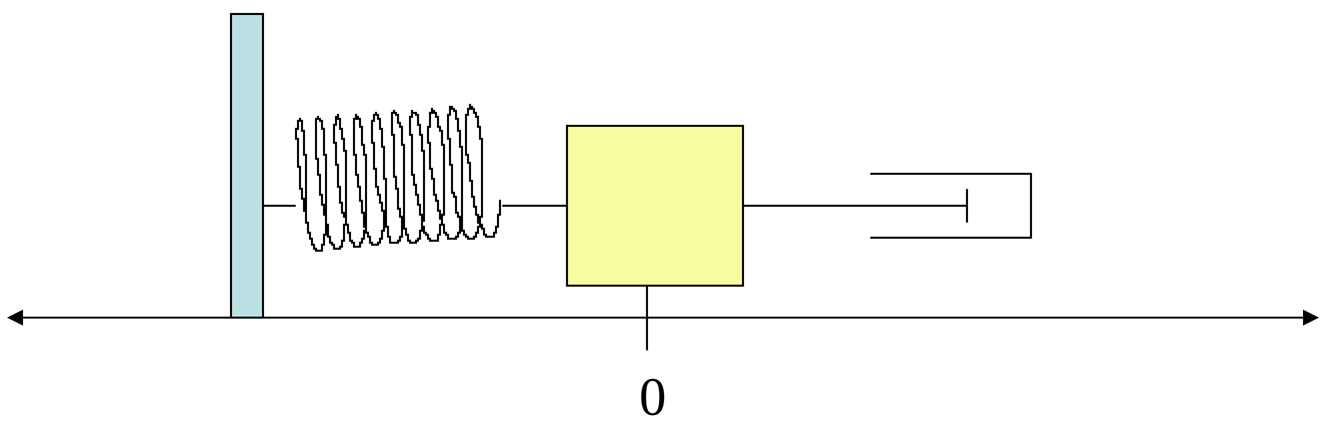
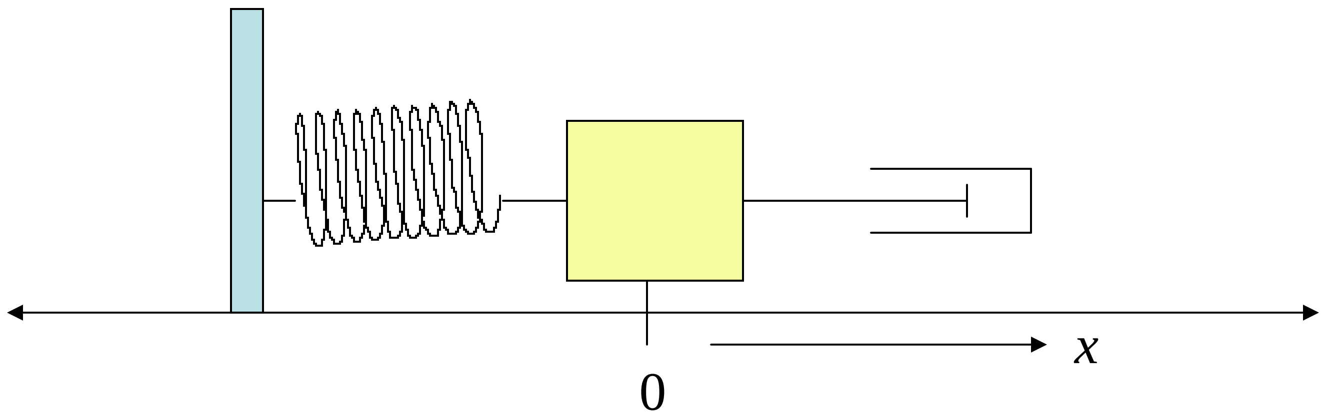
Springs, Dashpots, and the Law of Newton

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Suppose we have a spring, a mass, and a dashpot.

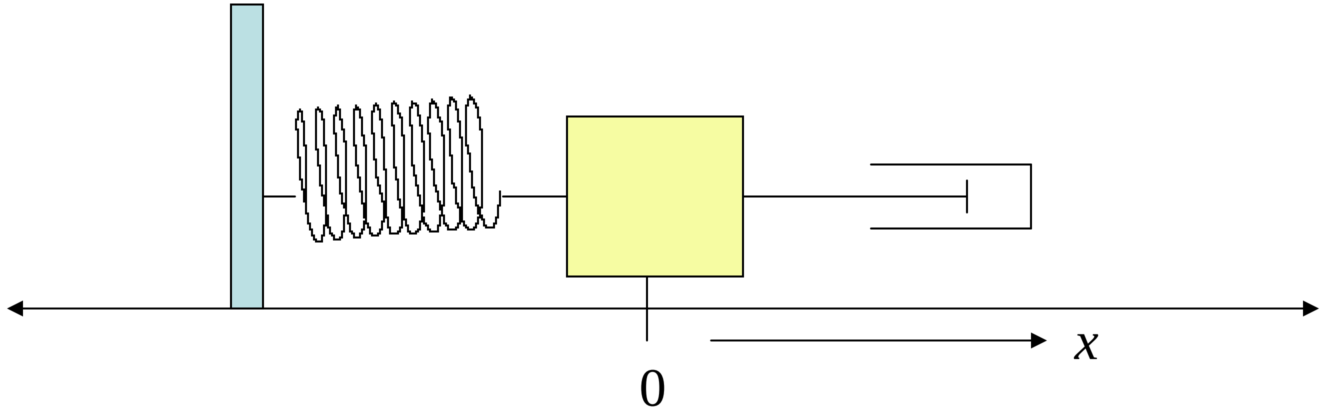


According to Hooke's Law, if we pull the mass *x* units to the right, then there is a restorative force in the the opposite direction equal to *–kx* where *k>0*.



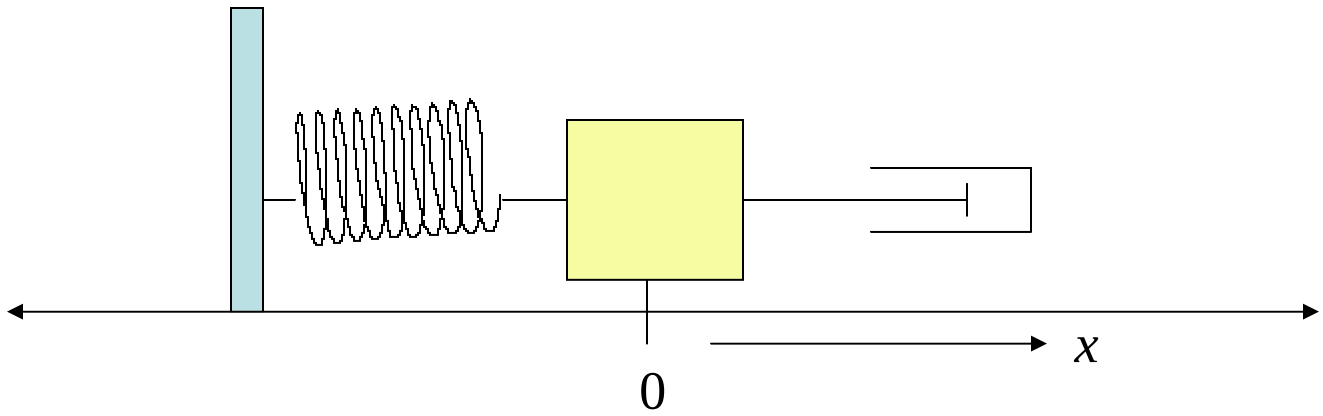
$$F = -kx$$

The dashpot is a dampening device such as found on a typical building door. It's effect is proportional to the velocity of the mass over time, and we can express that as below:



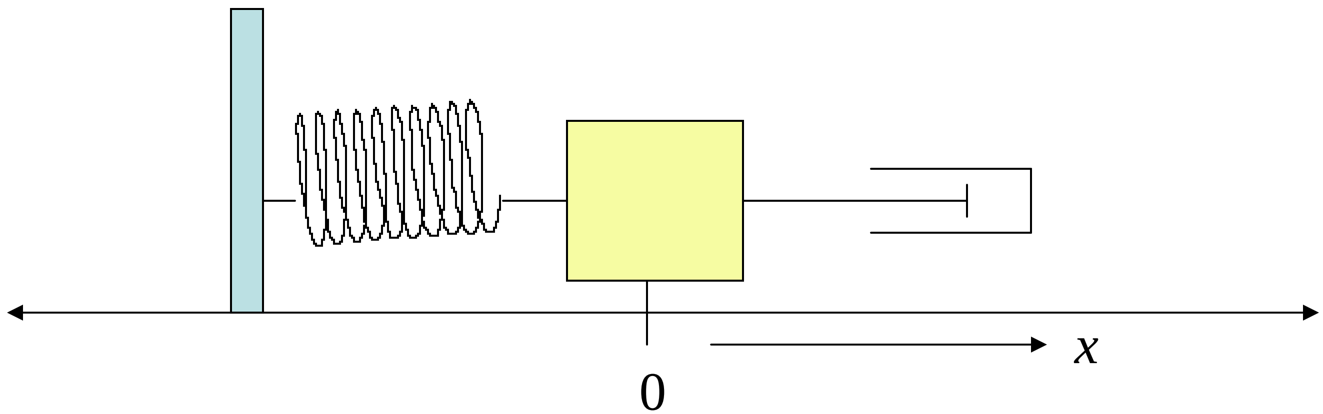
$$F = -kx - c\frac{dx}{dt} \quad (c > 0)$$

Finally, Newton's Second Law of Motion tells us that *force=mass x acceleration.* However, acceleration is the second derivative of position, and so we can write the left-hand side of the equation as follows:



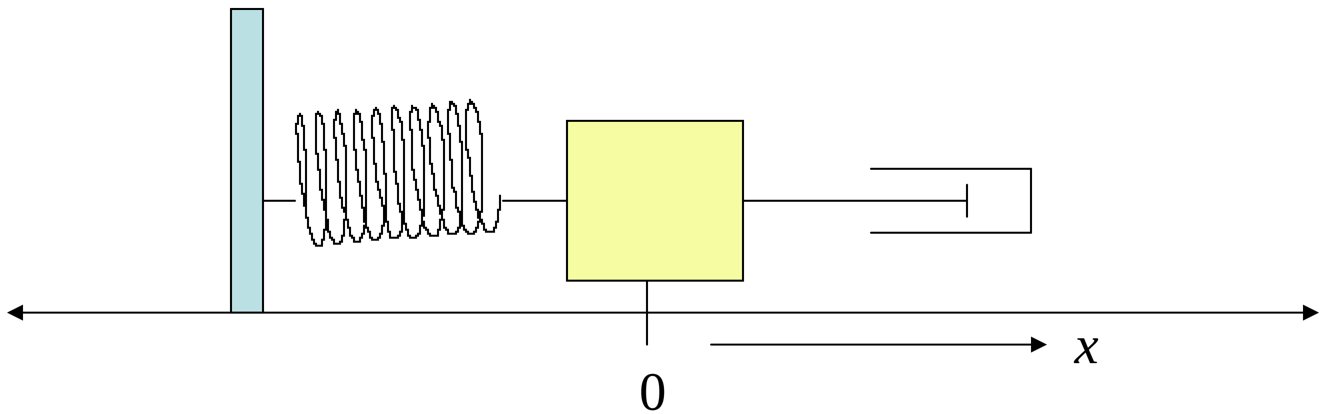
$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}$$

The end result is a second degree homogeneous linear differential equation with constant coefficients.



$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0$$

What follows are some initial value problems with graphs of the solutions.

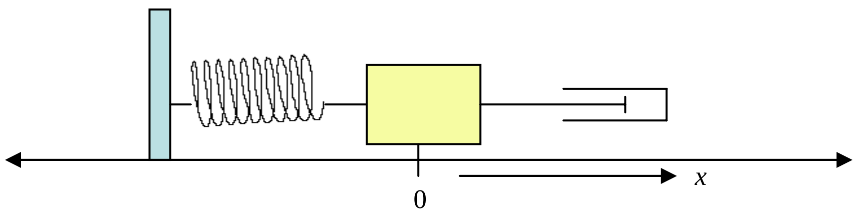


$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0$$

Example 1:

$$\frac{d^2x}{dt^2}+5\frac{dx}{dt}+6x=0$$

$$x(0)=3, \quad x'(0)=-8$$



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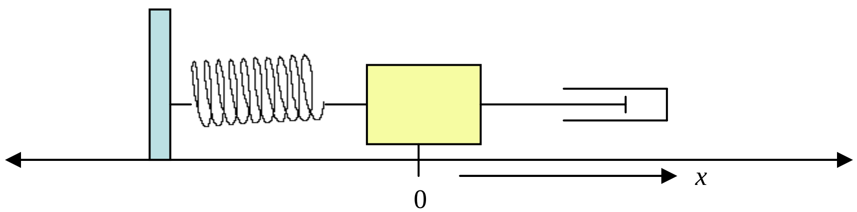

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$$

$$x(0) = 3, \quad x'(0) = -8$$

$$r^2 + 5r + 6 = 0$$

$$r_1 = -2, \quad r_2 = -3$$

$$x(t) = C_1e^{-2t} + C_2e^{-3t}$$

$$x'(t) = -2C_1e^{-2t} - 3C_2e^{-3t}$$

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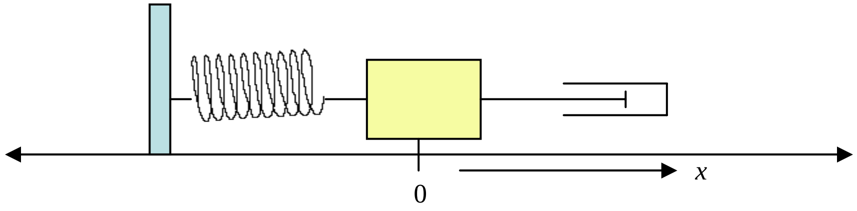

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$$

$$x(0) = 3, \quad x'(0) = -8$$

$$x(0) = C_1e^0 + C_2e^0 = C_1 + C_2 = 3$$

$$x'(0) = -2C_1e^0 - 3C_2e^0 = -2C_1 - 3C_2 = -8$$

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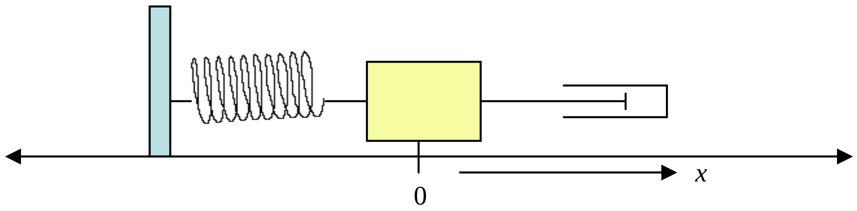

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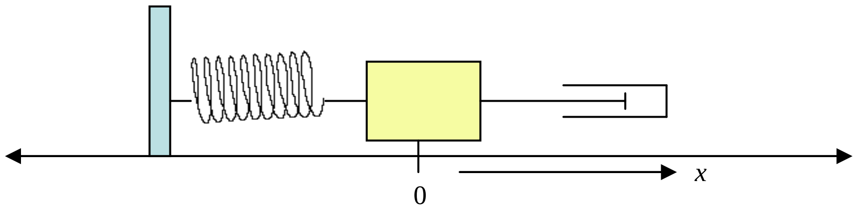
$$x(t) = C_1e^{-2t} + C_2e^{-3t}$$

$$x'(t) = -2C_1e^{-2t} - 3C_2e^{-3t}$$

$$C_1 = 1, \quad C_2 = 2$$

$$x(t) = e^{-2t} + 2e^{-3t}$$

Example 1:



$$\frac{d^2x}{dt^2}+5\frac{dx}{dt}+6x=0$$

$$x(0)=3,\quad x'(0)=-8$$

$$x(0)=C_1e^0+C_2e^0=C_1+C_2=3$$

$$x'(0)=-2C_1e^0-3C_2e^0=-2C_1-3C_2=-8$$

$$r^2+5r+6=0$$

$$r_1=-2,\quad r_2=-3$$

$$x(t)=C_1e^{-2t}+C_2e^{-3t}$$

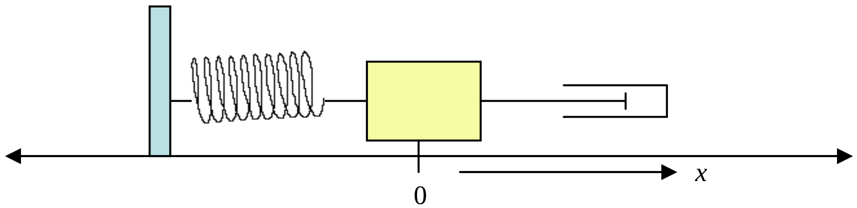
$$x'(t)=-2C_1e^{-2t}-3C_2e^{-3t}$$

$$x(t)=e^{-2t}+2e^{-3t}$$

Example 2:

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$$

$$x(0) = 3, \quad x'(0) = -8$$



HAVE FUN!