SET THEORY CONTINUED – PRACTICE

In set theory, the size or number of elements in a set is called its *cardinality*. There are various symbols that are used for *cardinality*, but my favorite is to simply enclose the set inside a pair of absolute value signs. Thus, if $A = \{a, b, c\}$, then $|A| = 3$ since the set has three elements. We can show that two sets have the same *cardinality* by finding a function that establishes a one-to-one correspondence between the elements of the sets. A function $f : A \to B$ is *one-to-one* if $\forall x, y \in A,\ x \neq y \Rightarrow f(x) \neq f(y)$. We also call a *one-to-one* function an *injection* or *injective function*. A function $f : A \to B$ is *onto* if $\forall y \in B,\ \exists x \in A \text{ such that } f(x) = y$. We also call an *onto* function a *surjection* or *surjective function*. We can now define a *one-to-one correspondence* between two sets *A* and *B* as a function $f : A \to B$ that is both *one-to-one* and *onto*. We also call a *one-to-one and onto* function a *bijection* or *bijective function*.

When we are dealing with *cardinality* or size of sets, everything behaves as we expect when the sets are finite. However, if our sets are infinite, then strange things can happen. For example, one set can be a proper subset of another, and yet the two sets can be the same size ($A \subset B$ & $|A| = |B|$). Additionally, some infinite sets can have more elements in them than other infinite sets (As you'll prove below, if $\mathbb{N} = \text{counting or natural numbers}$ and $\mathbb{R} = \text{real numbers}$, then $|\mathbb{N}| < |\mathbb{R}|$.). Note that if there is an *injective function* $f : A \to B$, but no *surjective function* $f : A \to B$, then we'll say that $|A| < |B|$.

1. If a set *A* has the same cardinality as the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, then we say that *A* is *countable*. Prove that the set of even natural numbers, $2\mathbb{N} = \{2, 4, 6, \ldots\}$, is *countable* by finding a *bijective function* $f : \mathbb{N} \to 2\mathbb{N}$. Conclude that there are just as many even natural numbers as natural numbers.

2. Prove that there are just as many numbers in the interval $(0,1)$ as there are real numbers by finding a *bijective function* $f : (0,1) \to \mathbb{R}$. (HINT: You can find a *bijection* by modifying a well-known trigonometric function.)

3. Study *Cantor's Diagonal Theorem* and use the argument to prove that there is no bijection $f : \mathbb{N} \to (0,1)$.

4. *Cantor's Diagonal Theorem* is an example of *proof by contradiction*. In other words, we make an assumption, prove that that assumption leads to a contradiction, and then we conclude the opposite of our assumption. However, some mathematicians don't like proofs done by this method. Review the concepts introduced in Lesson 1, and explain why some people don't like this method.

5. Conclude from 2 & 3 above that $|\mathbb{N}| < |\mathbb{R}|$.