SET THEORY CONTINUED – ANSWERS

In set theory, the size or number of elements in a set is called its *cardinality*. There are various symbols that are used for *cardinality*, but my favorite is to simply enclose the set inside a pair of absolute value signs. Thus, if $A = \{a, b, c\}$, then $|A| = 3$ since the set has three elements. We can show that two sets have the same *cardinality* by finding a function that establishes a one-to-one correspondence between the elements of the sets. A function $f : A \to B$ is *one-to-one* if $\forall x, y \in A,\ x \neq y \Rightarrow f(x) \neq f(y)$. We also call a *one-to-one* function an *injection* or *injective function*. A function $f : A \to B$ is *onto* if $\forall y \in B,\ \exists x \in A \text{ such that } f(x) = y$. We also call an *onto* function a *surjection* or *surjective function*. We can now define a *one-to-one correspondence* between two sets *A* and *B* as a function $f : A \to B$ that is both *one-to-one* and *onto*. We also call a *one-to-one and onto* function a *bijection* or *bijective function*.

When we are dealing with *cardinality* or size of sets, everything behaves as we expect when the sets are finite. However, if our sets are infinite, then strange things can happen. For example, one set can be a proper subset of another, and yet the two sets can be the same size ($A \subset B$ & $|A| = |B|$). Additionally, some infinite sets can have more elements in them than other infinite sets (As you'll prove below, if $\mathbb{N} = \text{counting or natural numbers}$ and $\mathbb{R} = \text{real numbers}$, then $|\mathbb{N}| < |\mathbb{R}|$.). Note that if there is an *injective function* $f : A \to B$, but no *surjective function* $f : A \to B$, then we'll say that $|A| < |B|$.

1. If a set *A* has the same cardinality as the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, then we say that *A* is *countable*. Prove that the set of even natural numbers, $2\mathbb{N} = \{2, 4, 6, \ldots\}$, is *countable* by finding a *bijective function* $f : \mathbb{N} \to 2\mathbb{N}$. Conclude that there are just as many even natural numbers as natural numbers.

   <u>Proof:</u> Define $f : \mathbb{N} \to 2\mathbb{N}$ by $f(x) = 2x$. Then clearly $f : \mathbb{N} \to 2\mathbb{N}$ is one-to-one since $x_1 \neq x_2 \Rightarrow 2x_1 \neq 2x_2$. Also, $f : \mathbb{N} \to 2\mathbb{N}$ is onto since if $y \in 2\mathbb{N}$, then *y* is even which means that $y/2 \in \mathbb{N}$, and, hence, $f(y/2) = 2 \cdot y/2 = y$. Therefore, $f : \mathbb{N} \to 2\mathbb{N}$ is a bijection, and $|\mathbb{N}| = |2\mathbb{N}|$. □

2. Prove that there are just as many numbers in the interval $(0,1)$ as there are real numbers by finding a *bijective function* $f : (0,1) \to \mathbb{R}$. (HINT: You can find a *bijection* by modifying a well-known trigonometric function.)

   <u>Proof:</u> Clearly, $g : (-\pi/2, \pi/2) \to \mathbb{R}$ defined by $g(x) = \tan x$ is a bijection. Thus, all we need to do is transform this function into a related function with domain $(0,1)$. If we replace $\tan x$ by $\tan \pi x$, then this will shift our interval $(-\pi/2, \pi/2)$ to $(-1/2, 1/2)$. Next, if we replace *x* by $x - 1/2$, then that will shift $(-1/2, 1/2)$ to $(0,1)$. Thus, $f : (0,1) \to \mathbb{R}$ defined by $f(x) = \tan \pi(x - 1/2)$ is a bijective function from $(0,1)$ to $\mathbb{R}$.

Therefore, $|(0,1)| = |\mathbb{R}|$. □

3. Study *Cantor's Diagonal Theorem* and use the argument to prove that there is no bijection $f:\mathbb{N}\to(0,1)$.

   Prove: There is no bijection $f:\mathbb{N}\to(0,1)$.

   Proof: By way of contradiction, suppose $f:\mathbb{N}\to(0,1)$ is a bijective function. Then the correspondence between elements of $\mathbb{N}$ and elements of $(0,1)$ could be written as a list like the following.

   $$\begin{array}{ccl} 1 & \to & 0.1451939\ldots \\ 2 & \to & 0.5876624\ldots \\ 3 & \to & 0.9942146\ldots \\ 4 & \to & 0.3621722\ldots \\ & \vdots & \end{array}$$

   Now, highlight the numbers along the diagonal from upper left on down.

   $$\begin{array}{ccl} 1 & \to & 0.\mathbf{1}451939\ldots \\ 2 & \to & 0.5\mathbf{8}76624\ldots \\ 3 & \to & 0.99\mathbf{4}2146\ldots \\ 4 & \to & 0.362\mathbf{1}722\ldots \\ & \vdots & \qquad\ddots \end{array}$$

   If we now generate a new number by changing each odd digit in the highlighted number to 2 and each even digit to 1, then our new number will belong to the interval $(0,1)$, but our function will have no natural number assigned to it since the *nth* digit of the decimal expansion will always differ from that of the number in our list that has been assigned to the natural number *n*. In other words, using the list above we can begin to generate the number $0.2112\ldots$. This number, however, doesn't correspond to 1 because the first digit is different from what's in our list. Similarly, it doesn't correspond to 2, since the second digit is different from what's in our list, and so on and so on. Thus, given any table or list for our function, we can always construct a real number in the interval $(0,1)$ that has no natural number assigned to it. Therefore, there is no bijection $f:\mathbb{N}\to(0,1)$. □

4. *Cantor's Diagonal Theorem* is an example of *proof by contradiction*. In other words, we make an assumption, prove that that assumption leads to a contradiction, and then we conclude the opposite of our assumption. However, some mathematicians don't like proofs done by this method. Review the concepts introduced in Lesson 1, and explain why some people don't like this method.

When we do a proof by contradiction, we are assuming that either *A* is true or *not*-*A* is true. In other words, we are assuming the Law of the Excluded Middle, and this makes some mathematicians uneasy. Hence, direct proofs that do not require this assumption are generally considered superior to proofs by contradiction.

5. Conclude from 2 & 3 above that $|\mathbb{N}| < |\mathbb{R}|$.

   From 3 above, we know that $|\mathbb{N}| < |(0,1)|$, and from 2 above we know that $|(0,1)| = |\mathbb{R}|$. Therefore, $|\mathbb{N}| < |\mathbb{R}|$.