SET THEORY - PRACTICE

In general, a mathematical proof is a convincing argument conforming to standard rules of logic that begins with a premise and ends with a conclusion. In the good ol' days (back in the seventies when I was young) there was a tendency to use as much mathematical shorthand notation as possible. In particular, two symbols from logic known, respectively, as the *universal quantifier* ($\forall$, *for every* …) and the *existential quantifier* ($\exists$, *there exists* …) were frequently employed as well as the symbol $\therefore$ for *therefore*. These days, however, there is a greater tendency to write proofs in plain English and to use the shorthand symbols a little more sparingly.

When you write a proof, think in terms of trying to write a convincing argument that a colleague could easily understand. This also means that when professional research mathematicians are writing proofs to be read by other researchers, they can be very brief in their arguments. On the other hand, when one is writing a proof for someone with less training in formal mathematics, a little more verbosity is often needed in order to make the argument convincing.

Each branch of mathematics tends to have its own style and technique of doing proofs. In particular, in set theory if one is trying to show that for two sets *A* and *B* that $A = B$, then one generally utilizes the following: "THEOREM: If *A* and *B* are sets such that $A \subseteq B$ and $B \subseteq A$, then $A = B$." Hence, proofs involving the equality of two sets *A* and *B* generally take the following form:

> PROOF: Let $x \in A$. … Thus, $x \in B$ and, hence, $A \subseteq B$. Now let $x \in B$. … Thus, $x \in A$ and, hence, $B \subseteq A$. Therefore, $A = B$. □
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> (NOTE: Mathematicians used to end their proofs with the letters *QED* which stands for *quod erat demonstrandum*, which means *that which was to be demonstrated*. However, a twentieth century mathematician named Paul Halmos felt it was a little presumptuous to always assume that one's proof was correct, and he introduced the practice of using a square (usually shaded) to indicate the end of a proof.)

Before we continue, here are a few basic definitions regarding set notation.

$\in$ - is an element of

U - The *universal set*. Whatever our universe of discourse is, i.e. real numbers, complex numbers, etc.

$\varnothing$ - The *null* or *empty set*. The set containing no elements.

$A \bigcup B = \{x \in U \mid x \in A \text{ or } x \in B\}$ (This is read as "*A* union *B*.")

$A \bigcap B = \{x \in U \mid x \in A \text{ and } x \in B\}$ (This is read as "*A* intersect *B*.")

$A' = \{x \in U \mid x \notin A\}$ (This is read as "*A*-complement.")

$A \subseteq B$ if and only if $\forall x \in A, x \in B$ (This is read as "*A* is a subset of *B*.")

$A \subset B$ if and only if $\forall x \in A, x \in B$ and $\exists y \in B$ such that $y \notin A$ (This is read as "*A* is a proper subset of *B*.")

The cardinality of a set *A* is the number of elements in *A*, and this is denoted by $|A|$. For example, if $A = \{a, b, c\}$, then $|A| = 3$. Also, $|\varnothing| = 0$.

1. Prove De Morgan's Laws.
   a. PROVE: $(A \cup B)' = A' \cap B'$.
   b. PROVE: $(A \cap B)' = A' \cup B'$

2. PROVE: $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

3. Explain why the *null set* is a subset of every set.

4. List the subsets of the following sets: $\varnothing, \{\varnothing\}, \{a\}. \{a, b\}, \{a, b, c\}$. Do you see a pattern with respect to the number of subsets?

5. Who was Georg Cantor? How did he live? How did he die?

6. Explain *Russell's Paradox*. What does it tell us about set theory? How do mathematicians "weasel out" of this paradox?