SET THEORY – ANSWERS

In general, a mathematical proof is a convincing argument conforming to standard rules of logic that begins with a premise and ends with a conclusion. In the good ol' days (back in the seventies when I was young) there was a tendency to use as much mathematical shorthand notation as possible. In particular, two symbols from logic known, respectively, as the *universal quantifier* ($\forall$, *for every* …) and the *existential quantifier* ($\exists$, *there exists* …) were frequently employed as well as the symbol $\therefore$ for *therefore*. These days, however, there is a greater tendency to write proofs in plain English and to use the shorthand symbols a little more sparingly.

When you write a proof, think in terms of trying to write a convincing argument that a colleague could easily understand. This also means that when professional research mathematicians are writing proofs to be read by other researchers, they can be very brief in their arguments. On the other hand, when one is writing a proof for someone with less training in formal mathematics, a little more verbosity is often needed in order to make the argument convincing.

Each branch of mathematics tends to have its own style and technique of doing proofs. In particular, in set theory if one is trying to show that for two sets *A* and *B* that $A = B$, then one generally utilizes the following: "THEOREM: If *A* and *B* are sets such that $A \subseteq B$ and $B \subseteq A$, then $A = B$." Hence, proofs involving the equality of two sets *A* and *B* generally take the following form:

> PROOF: Let $x \in A$. … Thus, $x \in B$ and, hence, $A \subseteq B$. Now let $x \in B$. Thus, $x \in A$ and, hence, $B \subseteq A$. Therefore, $A = B$. □
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> (NOTE: Mathematicians used to end their proofs with the letters *QED* which stands for *quod erat demonstrandum*, which means *that which was to be demonstrated*. However, a twentieth century mathematician named Paul Halmos felt it was a little presumptuous to always assume that one's proof was correct, and he introduced the practice of using a square (usually shaded) to indicate the end of a proof.)

Before we continue, here are a few basic definitions regarding set notation.

$\in$ - is an element of

U - The *universal set*. Whatever our universe of discourse is, i.e. real numbers, complex numbers, etc.

$\varnothing$ - The *null* or *empty set*. The set containing no elements.

$A \bigcup B = \{x \in U \mid x \in A \text{ or } x \in B\}$ (This is read as "*A* union *B*.")

$A \bigcap B = \{x \in U \mid x \in A \text{ and } x \in B\}$ (This is read as "*A* intersect *B*.")

$A' = \{x \in U \mid x \notin A\}$ (This is read as "*A*-complement.")

$A \subseteq B$ if and only if $\forall x \in A, x \in B$ (This is read as "*A* is a subset of *B*.")

$A \subset B$ if and only if $\forall x \in A, x \in B$ and $\exists y \in B$ such that $y \notin A$ (This is read as "*A* is a proper subset of *B*.")
The cardinality of a set *A* is the number of elements in *A*, and this is denoted by $|A|$. For example, if $A = \{a, b, c\}$, then $|A| = 3$. Also, $|\varnothing| = 0$.

1. Prove De Morgan's Laws.
   a. PROVE: $(A \cup B)' = A' \cap B'$.

      PROOF: Let $x \in (A \cup B)'$. Then $x \notin A$ and $x \notin B$ (since, otherwise, we would have $x \in (A \cup B)$). Thus, if $x \notin A$ and $x \notin B$, then $x \in A'$ and $x \in B'$ which implies that $x \in A' \cap B'$. Therefore, $(A \cup B)' \subseteq A' \cap B'$.

      Now suppose that $x \in A' \cap B'$. Then $x \in A'$ and $x \in B'$ which implies that $x \notin A$ and $x \notin B$, and, hence, $x \notin A \cup B$. Therefore, $x \in (A \cup B)'$, and, thus, $A' \cap B' \subseteq (A \cup B)'$. Furthermore, since $(A \cup B)' \subseteq A' \cap B'$ and $A' \cap B' \subseteq (A \cup B)'$, it now follows that $(A \cup B)' = A' \cap B'$. □

   b. PROVE: $(A \cap B)' = A' \cup B'$

      PROOF: Let $x \in (A \cap B)'$. Then $x \notin A$ or $x \notin B$ (since, otherwise, we would have $x \in (A \cap B)$). Thus, if $x \notin A$ or $x \notin B$, then $x \in A'$ or $x \in B'$ which implies that $x \in A' \cup B'$. Therefore, $(A \cap B)' \subseteq A' \cup B'$.

      Now suppose that $x \in A' \cup B'$. Then $x \in A'$ or $x \in B'$ which implies that $x \notin A$ or $x \notin B$, and, hence, $x \notin A \cap B$. Therefore, $x \in (A \cap B)'$, and, thus, $A' \cup B' \subseteq (A \cap B)'$. Furthermore, since $(A \cap B)' \subseteq A' \cup B'$ and $A' \cup B' \subseteq (A \cap B)'$, it now follows that $(A \cap B)' = A' \cup B'$. □

2. PROVE: $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

   PROOF: Suppose $x \in (A \cup B) \cap C$. Then $x \in A \cup B$ and $x \in C$. However, if $x \in A \cup B$, then $x \in A$ or $x \in B$. If $x \in A$, then $x \in A \cap C$, and if $x \in B$, then $x \in B \cap C$. Thus, one way or another, we have that $x \in (A \cap C) \cup (B \cap C)$ and, hence, $(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C)$.

   Now suppose that $x \in (A \cap C) \cup (B \cap C)$. Then $x \in A \cap C$ or $x \in B \cap C$, and, hence, it follows that $x \in C$. It also follows that $x \in A$ or $x \in B$, and thus, $x \in A \cup B$. Consequently, $x \in (A \cup B) \cap C$, and, hence, $(A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap C$.

   $\therefore (A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. □

3. Explain why the *null set* is a subset of every set.

   By definition, the *null set* is the set containing no objects. Also, $A \subseteq B$ means that if $x \in A$, then $x \in B$. If we replace *A* by $\varnothing$, then we can rewrite this condition as $\varnothing \subseteq B$ means that if $x \in \varnothing$ then $x \in B$ (or in other words, $x \in \varnothing \Rightarrow x \in B$). However, the statement $x \in \varnothing$ is always false, and recall that a false statement can imply anything. Hence, it is true that $x \in \varnothing \Rightarrow x \in B$, and, therefore, $\varnothing \subseteq B$ for any set *B*.

4. List the subsets of the following sets: $\varnothing, \{a\}, \{a,b\}, \{a,b,c\}$. Do you see a pattern with respect to the number of subsets?

   We'll denote the set of all subset of a set *A* by $P(A)$. Consequently,

   $P(\varnothing) = \{\varnothing\}$

   $P(\{a\}) = \{\varnothing, \{a\}\}$

   $P(\{a,b\}) = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$

   $P(\{a,b,c\}) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

   If we now denote the number of elements in a set *A* by $|A|$, then

   $|P(\varnothing)| = 2^0 = 1$

   $|P(\{a\})| = 2^1 = 2$

   $|P(\{a,b\})| = 2^2 = 4$

   $|P(\{a,b,c\})| = 2^3 = 8$

   These results correctly suggest that if a finite set *A* has *n* elements, then $|P(A)| = 2^n$.

5. Who was Georg Cantor? How did he live? How did he die?

   George Cantor was born in 1845 and died in 1918, and he is best remembered as the creator of set theory. While today we consider set theory to be the foundation for all of mathematics, in his day Cantor and his work were the subject of several vicious attacks from not only his former teacher, Leopold Kronecker, but also other leading mathematicians of the day. This is because Cantor's work included the existence of infinities of different sizes as well as various other items that had not been a part of mathematics up to that point. For example, prior to Cantor, infinity was a subject to be eschewed in mathematics as is evidenced by the following quote from mathematician Carl Friedrich Gauss:

   "I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking, the true meaning being a limit which certain ratios approach indefinitely close, while others are permitted to increase without restriction."

Because his ideas were so radical at the time, his former professor, Kronecker, actively worked to block the publication of his papers and appointments to more prestigious institutions. These attacks on Cantor appear to have eventually taken their toll, and Cantor suffered his first hospitalization for chronic depression in 1884. He was to be hospitalized many more times in his life for this mental disorder including the last year of his life which he spent in a sanatorium in Halle, Germany. In spite of the opposition generated during Cantor's lifetime, his theories eventually gained wide acceptance throughout mathematics, and in the first half of the twentieth century, the great mathematician David Hilbert declared, "No one shall expel us from the Paradise that Cantor has created."

6. Explain *Russell's Paradox*. What does it tell us about set theory? How do mathematicians "weasel out" of this paradox?

   At its beginning, the word *set* seemed to be just another word for *collection*, and surely we can talk about the collection of anything we wish. Or so it seemed. This point of view that we naturally understand what a set is without needing any formal axiomatic structure is now referred to as *naive set theory*, and its limitations were discovered in 1901 when Bertrand Russell formulated what we now call *Russell's Paradox*. The heart of the paradox is that, intuitively, a set *R* can either be a member or element of itself or, alternatively, not an element of itself. Most of time we experience the latter. For example, if $A=\{1,2\}$, then $1\in A$ and $2\in A$, but $A\notin A$ (even though $A\subseteq A$). However, on the other hand, if I say, "Let *A* be the set of all sets that I can describe with a finite number of words," then it seems like I have described the set *A* itself with a finite number of words, and, hence, $A\in A$. In this spirit, Bertrand Russell asked us to consider the set *R* defined as the set of all sets that do not contain themselves as members. In other words, $R=\{A\,|\,A\notin A\}$. We now ask ourselves the question, "Is $R\in R$?" If $R\in R$, then by definition, $R\notin R$, and if $R\notin R$, then again, by definition, $R\in R$. Thus, we arrive at the following paradox, $R\in R\Leftrightarrow R\notin R$.

   Russell's Paradox showed us that, if we wanted to avoid contradictions, we had to be much more careful about what we did and what we didn't call a *set*. The ultimate result, for most mathematicians, was the creation of an axiomatic version of set theory known as *Zermelo-Frankel*. This version of set theory allows infinite sets of all sorts to exist, but, at the same time, it doesn't allow us to talk about things that are so large or so self-referential that they lead us into paradoxes. Furthermore, in this axiomatic formulation, the term *set* is left undefined, and those collections that we don't believe as deserving to be called sets within the framework of *Zermelo-Frankel* we simply label as *classes*. Thus, the collection *R* defined above is simply referred to as a *class* instead of a *set*, and the paradox is avoided. Nonetheless, in my opinion, we haven't really resolved the paradox. We've simply decided to ignore an inconvenient truth and, instead, sweep it under the rug. Furthermore, we should note that paradoxes, in general, are quite interesting because they show us where our logical framework for reality tends to break down. More paradoxes to come!