Separable Differential Equations

$$P(x)+Q(y)\frac{dy}{dx}=0$$

The easiest kind of first order differential equation to solve is a separable differential equation such as the one below.

$$\frac{dy}{dx} = y$$

An equation is separable if we can separate the variables so that everything with x is on one side and everything with y is on the other side.

$$\frac{dy}{dx} = y$$

$$\frac{1}{y}dy = dx \qquad \text{(differential form)}$$

Now, we integrate both sides.

$$\int \frac{1}{y} dy = \int dx$$

If it doesn’t look quite kosher to you to be integrating one side with respect to y and another with respect to x, then use integration by substitution to justify your equation.

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$$\int \frac{1}{y}\,dy=\int \frac{1}{y}\frac{dy}{dx}\,dx=\int dx$$

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$$\int \frac{1}{y}\,dy = \lim_{\Delta y \to 0} \sum \frac{1}{y}\Delta y = \lim_{\Delta x \to 0} \sum \frac{1}{y}\frac{\Delta y}{\Delta x}\Delta x = \int \frac{1}{y}\frac{dy}{dx}\,dx = \int dx$$

Now let's do the problem:

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$$\frac{dy}{dx} = y$$

$$y = \pm Ce^{x} = Ce^{x}$$

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And what do we do if we have an initial value problem?

$$\frac{dy}{dx} = y, \quad y(0) = 5$$

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$$y = 5e^{x}$$

And now, here's another example:

$$\frac{2y}{y^2+5}\frac{dy}{dx}=\frac{1}{x^2}$$

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$$\ln(y^2+5)=-\frac{1}{x}+c$$

$$\frac{2y}{y^2+5}\frac{dy}{dx}=\frac{1}{x^2}$$

$$y^2+5=e^{-\frac{1}{x}+c}=e^c e^{-\frac{1}{x}}=Ce^{-\frac{1}{x}}$$

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$$y=\pm\sqrt{Ce^{-\frac{1}{x}}-5},\quad C>0$$

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$$\frac{2y}{y^2+5}\frac{dy}{dx} = \frac{2\sqrt{Ce^{-\frac{1}{x}} - 5}}{Ce^{-\frac{1}{x}}} \cdot \frac{1}{2}\left(Ce^{-\frac{1}{x}} - 5\right)^{-\frac{1}{2}} \cdot Ce^{-\frac{1}{x}} \cdot \frac{1}{x^2} = \frac{1}{x^2}$$

Check 2: $y = -\sqrt{Ce^{-\frac{1}{x}} - 5}$

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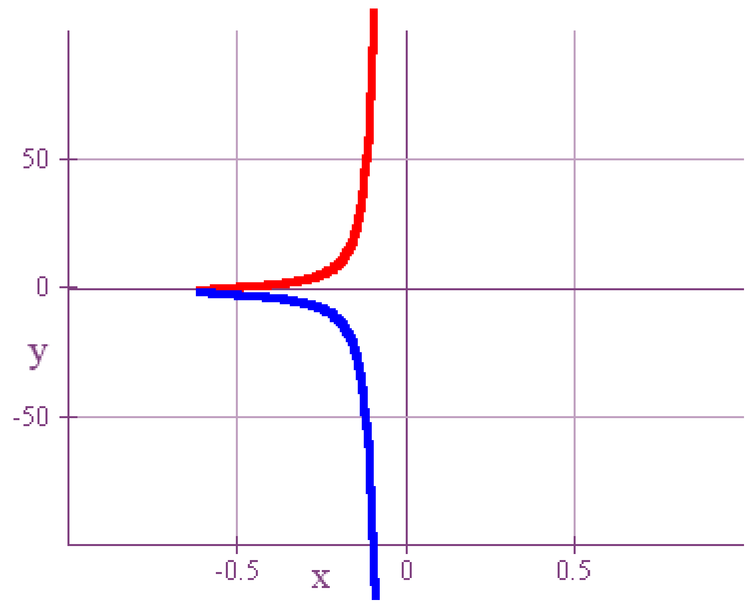
$$\frac{2y}{y^2+5}\frac{dy}{dx} = \frac{-2\sqrt{Ce^{-\frac{1}{x}} - 5}}{Ce^{-\frac{1}{x}}} \cdot \left(-\frac{1}{2}\right)\left(Ce^{-\frac{1}{x}} - 5\right)^{-\frac{1}{2}} \cdot Ce^{-\frac{1}{x}} \cdot \frac{1}{x^2} = \frac{1}{x^2}$$

Graphs with $C = 1$:



$$y = \sqrt{e^{-\frac{1}{x}} - 5}$$

$$y = -\sqrt{e^{-\frac{1}{x}} - 5}$$