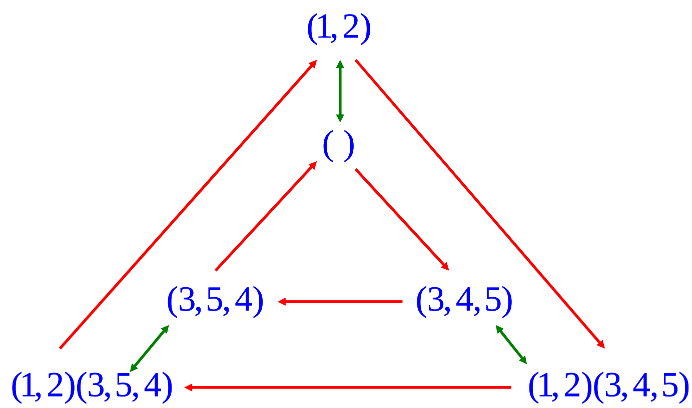
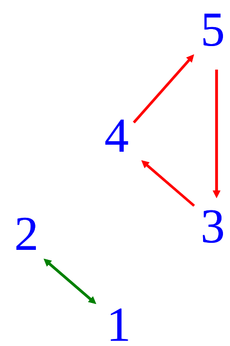
Lesson 19

SEMIDIRECT PRODUCTS

Let's begin by revisiting the Cayley graph for the direct product of the cyclic group of order 2 with the cyclic group of order 3, $C_2 \times C_3$.

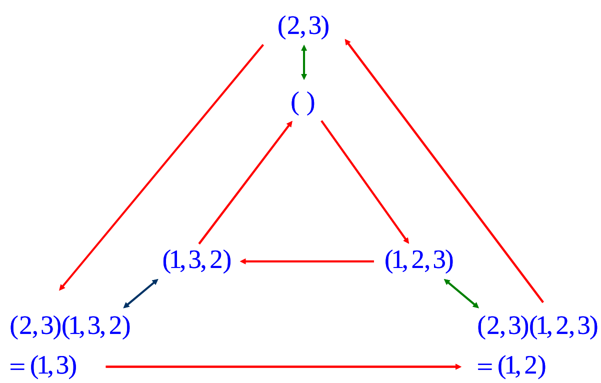


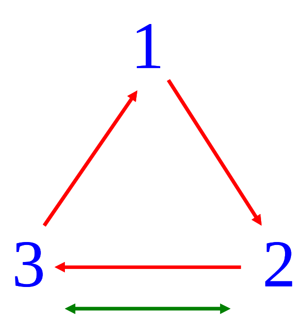
And let's also examine the corresponding generator diagram.



From the above, it should be clear that the cycle of length 2 that moves the numbers 1 & 2 operates independently of the cycle of length 3 that moves the numbers 3, 4, & 5. In other words, these two cycles commute with one another. In cycle notation we can write this as $(1,2)(3,4,5)=(3,4,5)(1,2)$. Since our direct product is generated by the cycles $(1,2)$ and $(3,4,5)$ and since these cycles commute with one another, it's going to follow that C_2 and C_3 are both normal subgroups of $C_2 \times C_3$. Let's give an example where we will let $a=(1,2)(3,4,5)(3,4,5)=(1,2)(3,5,4)$ be a typical element of our direct product. Then $a^{-1}=(4,5,3)(2,1)$. Also, $a^{-1}C_3a=(4,5,3)(2,1)C_3(1,2)(3,5,4)=(4,5,3)\left[(2,1)C_3(1,2)\right](3,5,4)$. Now since $(2,1)$ & $(1,2)$ commute with the elements in C_3, it follows that $(2,1)C_3(1,2)=C_3$. Hence, $(4,5,3)\left[(2,1)C_3(1,2)\right](3,5,4)=(4,5,3)C_3(3,5,4)$. But now, since $(4,5,3)$ & $(3,5,4)$ are elements of C_3, we also have that $(4,5,3)\left[(2,1)C_3(1,2)\right](3,5,4)=(4,5,3)C_3(3,5,4)=C_3$. This same argument applies to any element of $C_2 \times C_3$ that is written as a product of the cycles $(1,2)$ and $(3,4,5)$, and that is why C_3 is a normal subgroup of $C_2 \times C_3$ and, similarly, why C_2 is a normal subgroup of $C_2 \times C_3$. And when we look at the Cayley graph above, we see that applying the "flip" $(1,2)$ to elements doesn't change the direction of any of the red arrows. In other words, our red arrows are cycling in a clockwise direction whether we do the flip or not. In summary, we can say that C_2 and C_3 are both normal subgroups of $C_2 \times C_3$, the intersection of C_2 and C_3 is the identity, ( ), and the product of all elements in C_2 with elements in C_3 is $C_2 \cdot C_3 \cong C_2 \times C_3$. Recall that the elements in C_2 and C_3 generate $C_2 \times C_3$, and since these elements commute with one another, we can write every element in $C_2 \times C_3$ essentially as a product of an element from C_2 with an element from C_3. Hence, $C_2 \cdot C_3 \cong C_2 \times C_3$. By the way, our statement involving two normal subgroups whose intersection is the identity and whose product is the whole group is often taken as the definition of the direct product in many abstract algebra textbooks.

Now let's look at what happens when we have two subgroups whose intersection is the identity and whose product is the entire group, but only one of the subgroups is normal. An example of this can be found in D_3, the dihedral group of degree 3 that describes the symmetry of an equilateral triangle. Below are a Cayley graph and a generator diagram for this group.





We can see from these diagrams that the cycles $(1,2,3)$ and $(2,3)$ generate this group. Also, the subgroup generated by $(2,3)$ is $C_2 = \{(\ ),(2,3)\}$ and the subgroup generated by $(1,2,3)$ is $C_3 = \{(\ ),(1,2,3),(1,3,2)\}$. Clearly, the intersection of these two subsets is the identity, and the Cayley graph above illustrates that every element in the group can be written as a product of an element from $C_2 = \{(\ ),(2,3)\}$ with an element from $C_3 = \{(\ ),(1,2,3),(1,3,2)\}$. Hence, our group is equal to $C_2 \cdot C_3$. But, are either of these subgroups normal subgroups? It turns out that $C_3 = \{(\ ),(1,2,3),(1,3,2)\}$ is a normal subgroup, but $C_2 = \{(\ ),(2,3)\}$ isn't since for $a = (1,2)$, $a^{-1}(2,3)a = (2,1)(2,3)(1,2) = (1,3)$ is not an element of $C_2 = \{(\ ),(2,3)\}$. Thus, since only one of our two subgroups is a normal subgroup, the larger group $C_2 \cdot C_3$ is not isomorphic to the direct product of C_2 & C_3, but since one of these subgroups is normal and since their intersection is the identity and their product is the entire group, we get what we call the semidirect product of the normal subgroup C_3 by the non-normal subgroup C_2, and we write this as $C_3 >\!\!\lhd C_2$. And we can see a distinct difference between $C_3 \times C_2$ and $C_3 >\!\!\lhd C_2$ when we look at their Cayley gaphs. In the direct product, the elements of the two subgroups commute with one another, and our red arrows all go in the clockwise direction. However, in the Cayley graph for $C_3 >\!\!\lhd C_2$ things get twisty. Elements from the two subgroups do not always commute with one another, and our red arrows switch from clockwise to counterclockwise. Semidirect products always have this noncommutative twist that is lacking in the simpler direct products.