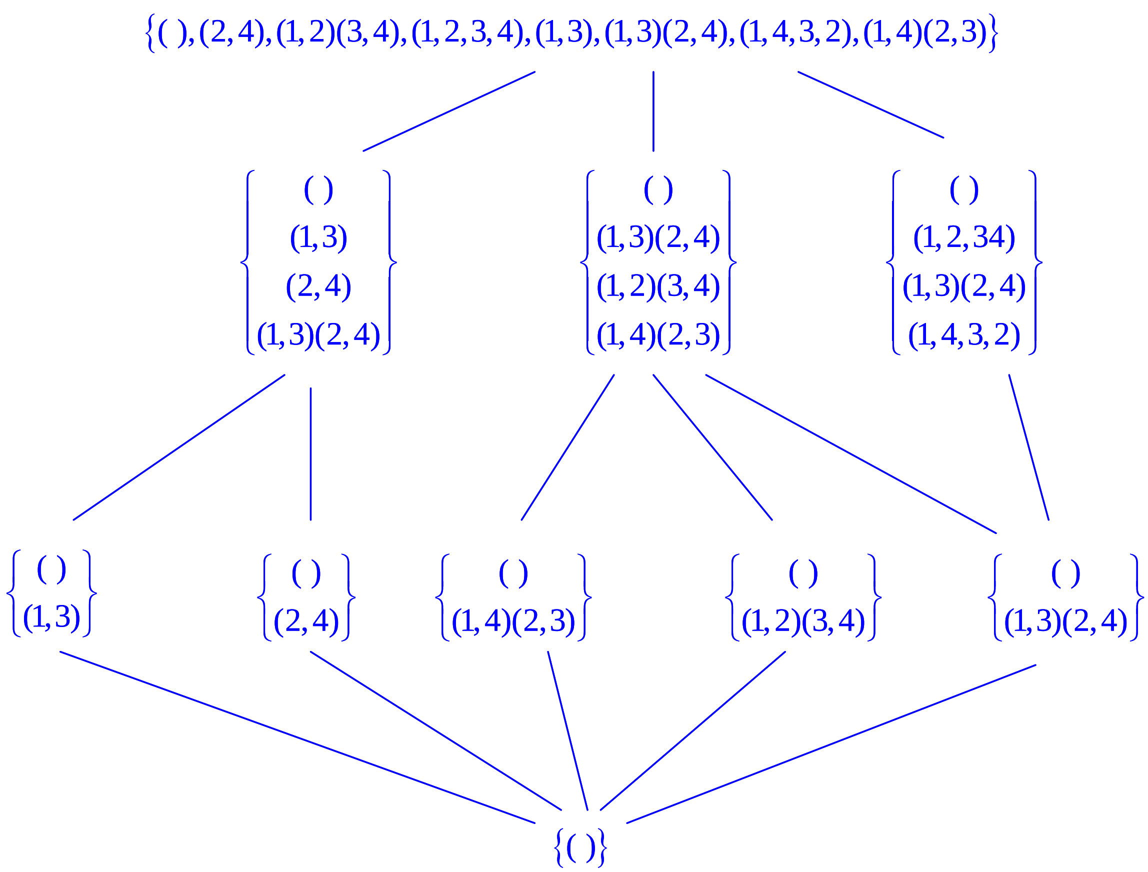
Lesson 19

SEMIDIRECT PRODUCTS – ANSWERS

Below is the subgroup lattice for D_4, the dihedral group of order 8 that is associated with the symmetries of a square.

D_4 = { (), (2,4), (1,2)(3,4), (1,2,3,4), (1,3), (1,3)(2,4), (1,4,3,2), (1,4)(2,3) }



Let $C_2 = \{( ), (1,3)\}$ and let $C_4 = \{( ), (1,2,3,4), (1,3)(2,4), (1,4,3,2)\}$.

1. Verify that $C_2 \cap C_3 = ( )$, that the identity is the only element in the intersection of the two subgroups.

   Direct examination of the elements of the two subgroups confirms that $C_2 \cap C_3 = ( )$.

2. Verify that $C_2 \cdot C_3 = D_4$, that the product of the two subgroups gives us back the entire group.

$$\begin{array}{ccc} (\,)(\,) & = & (\,) \\ (\,)(1,2,3,4) & = & (1,2,3,4) \\ (\,)(1,3)(2,4) & = & (1,3)(2,4) \\ (\,)(1,4,3,2) & = & (1,4,3,2) \\ (1,3)(\,) & = & (1,3) \\ (1,3)(1,2,3,4) & = & (1,4)(3,2) \\ (1,3)(1,3)(2,4) & = & (2,4) \\ (1,3)(1,4,3,2) & = & (1,2)(3,4) \end{array}$$

Therefore, $C_2 \cdot C_3 = D_4$.

3. Verify that $C_2 = \{(\,),(1,3)\}$ is not a normal subgroup of D_4.

If $a = (1,2)$, then $a^{-1}C_2 a = (1,2)\begin{Bmatrix} (\,) \\ (1,3) \end{Bmatrix}(1,2) = \begin{Bmatrix} (\,) \\ (2,3) \end{Bmatrix} \neq C_2$. Hence, C_2 is not a normal subgroup of D_4.

4. Verify that $C_4 = \{(\,),(1,2,3,4),(1,3)(2,4),(1,4,3,2)\}$ is a normal subgroup of D_4.

Every element of D_4 can be written as a product of an element in C_2 and an element in C_4. Hence, if *ab* is an element of D_4 where *a* is an element of C_2 and *b* is an element of C_4, then we need to argue that $(ab)^{-1}C_4(ab) = b^{-1}a^{-1}C_4 ab = C_4$. Well, it's clear that conjugation of C_4 by the identity gives us back C_4, so we can assume that $a = (1,3)$ and

$$(1,3)C_4(1,3) = (1,3)\begin{Bmatrix} (\,) \\ (1,2,3,4) \\ (1,3)(2,4) \\ (1,4,3,2) \end{Bmatrix}(1,3) = \begin{Bmatrix} (1,3)(\,)(1,3) \\ (1,3)(1,2,3,4)(1,3) \\ (1,3)(1,3)(2,4)(1,3) \\ (1,3)(1,4,3,2)(1,3) \end{Bmatrix} = \begin{Bmatrix} (\,) \\ (1,4,3,2) \\ (1,3)(2,4) \\ (1,2,3,4) \end{Bmatrix} = C_4.$$ And finally,

since *b* is an element of C_4, conjugation of C_4 by *b* must also give us back C_4. Hence, $(ab)^{-1}C_4(ab) = b^{-1}a^{-1}C_4 ab = C_4$, and C_4 is a normal subgroup of D_4.

5. Conclude that D_4 is isomorphic to the semidirect product of C_4 by C_2, $D_4 \cong C_4 >\!\!\lhd C_2$.

Since $C_2 \cap C_3 = (\,)$, $C_2 \cdot C_3 = D_4$, and C_4 is a normal subgroup of D_4 while C_2 isn't, it follows that D_4 is isomorphic to the semidirect product of C_4 by C_2, $D_4 \cong C_4 >\!\!\lhd C_2$.