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We are now going to examine not only some quotient groups, but also quotients of quotients and quotients of quotients in order to get a feel for what they are really like. Thus, let's start with the group  $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ . Then |G| = 16, and the elements of G are,

$$G = \begin{cases} (0,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (0,1,1,0), (0,1,0,1), (0,0,1,1), (0,1,1,1), \\ (1,0,0,0), (1,1,0,0), (1,0,1,0), (1,0,0,0), (1,1,1,0), (1,1,0,1), (1,0,1,1), (1,1,1,1) \end{cases}$$

This group is abelian, and so all of its subgroups are normal. In particular, let's consider the following subgroups:

$$\begin{split} N_1 = & \left\{ (0,0,0,0), \, (1,0,0,0) \right\} \\ N_2 = & \left\{ (0,0,0,0), \, (1,0,0,0), \, (0,1,0,0), \, (1,1,0,0) \right\} \\ N_3 = & \left\{ (0,0,0,0), \, (1,0,0,0), \, (0,1,0,0), \, (0,0,1,0), \, (1,0,1,0), \, (0,1,1,0), \, (1,1,1,0) \right\} \end{split}$$

Notice that  $N_1 \subset N_2 \subset N_3$ . Also,  $|G/N_1| = 8$ , and the cosets in  $G/N_1$  are,

$$G/N_1 = \begin{cases} \left\{ (0,0,0,0) \\ (1,0,0,0) \\ \end{array} \right\}, \left\{ (0,1,0,0) \\ (1,1,0,0) \\ \end{array}, \left\{ (0,0,1,0) \\ (1,0,1,0) \\ \end{array} \right\}, \left\{ (0,0,0,1) \\ (1,0,0,1) \\ \end{array}, \left\{ (0,1,1,0) \\ (1,1,1,0) \\ \end{array}, \left\{ (0,1,0,1) \\ (1,1,0,0) \\ \end{array}, \left\{ (0,0,1,1) \\ (1,0,1,1) \\ \end{array} \right\}, \left\{ (0,1,1,1) \\ (1,1,1,1) \\ \end{array} \right\}$$

Now we want to look at  $G/N_2$  which by our isomorphism theorem is isomorphic to  $(G/N_1)/(N_2/N_1)$ . Thus, we'll first write down the cosets for  $G/N_2$  and then compare this to the cosets in  $(G/N_1)/(N_2/N_1)$ . In particular,  $|G/N_2| = 4$ , and the cosets in  $G/N_2$  are,

$$G/N_2 = \left\{ \begin{cases} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \end{cases}, \begin{cases} (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{cases}, \begin{cases} (0,0,0,1) \\ (1,0,0,1) \\ (0,1,0,1) \\ (1,1,0,1) \end{cases}, \begin{cases} (0,0,1,1) \\ (1,0,0,1) \\ (0,1,0,1) \\ (1,1,1,1) \end{cases} \right\}$$

Now we want to compare this to the cosets in  $(G/N_1)/(N_2/N_1)$ , so let's first write down  $N_2/N_1$ .

$$N_2/N_1 = \left\{ \begin{cases} (0,0,0,0) \\ (1,0,0,0) \end{cases}, \begin{cases} (0,1,0,0) \\ (1,1,0,0) \end{cases} \right\}$$

Since,

$$G/N_1 = \begin{cases} \left\{ (0,0,0,0) \right\}, \left\{ (0,1,0,0) \right\}, \left\{ (0,0,1,0) \right\}, \left\{ (0,0,0,1) \right\}, \\ \left\{ (1,0,0,0) \right\}, \left\{ (1,1,0,0) \right\}, \left\{ (1,0,1,0) \right\}, \left\{ (1,0,0,1) \right\}, \\ \left\{ (0,1,1,0) \right\}, \left\{ (0,1,0,1) \right\}, \left\{ (0,0,1,1) \right\}, \left\{ (0,1,1,1) \right\}, \\ \left\{ (1,1,1,0) \right\}, \left\{ (1,1,0,0) \right\}, \left\{ (1,0,1,1) \right\}, \left\{ (1,1,1,1) \right\} \end{cases},$$

We have that,

$$(G/N_1)/(N_2/N_1) = \begin{cases} \left\{ \begin{cases} (0,0,0,0) \\ (1,0,0,0) \end{cases} \right\}, \\ \left\{ (0,1,0,0) \\ (1,1,0,0) \end{cases} \right\}, \\ \left\{ (0,1,0,0) \\ (1,1,0,0) \end{cases}, \\ \left\{ (0,1,1,0) \\ (1,1,0,0) \end{cases}, \\ \left\{ (0,1,0,1) \\ (1,1,0,1) \end{cases}, \\ \left\{ (0,1,0,1) \\ (1,1,0,1) \end{cases}, \\ \left\{ (0,1,0,1) \\ (1,1,0,1) \right\} \right\}, \\ \left\{ (0,1,1,1) \\ (1,1,1,1) \right\} \end{cases}$$

Notice now the structural similarity between  $(G/N_1)/(N_2/N_1)$  and  $G/N_2$ .

$$G/N_2 = \begin{cases} \begin{pmatrix} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \end{pmatrix}, \begin{pmatrix} (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{pmatrix}, \begin{pmatrix} (0,0,0,1) \\ (1,0,0,1) \\ (0,1,0,1) \\ (1,1,0,1) \end{pmatrix}, \begin{pmatrix} (0,0,1,1) \\ (1,0,0,1) \\ (1,1,0,1) \end{pmatrix}, \begin{pmatrix} (0,0,1,1) \\ (1,0,1,1) \\ (1,1,1,1) \end{pmatrix} \end{cases}$$

And now we'll move on to  $G/N_3$  which is isomorphic to both  $(G/N_2)/(N_3/N_2)$  and  $\left[(G/N_1)/(N_2/N_1)\right]/\left[(N_3/N_1)/(N_2/N_1)\right]$ . As before, we'll start with the simplest quotient group,  $G/N_3$ , and we'll methodically construct the other quotient groups so that we can observe the similarities. Thus,  $|G/N_3|=2$ , and the cosets in  $G/N_3$  are,

$$G/N_3 = \begin{cases} \begin{pmatrix} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \\ (0,0,1,0) \\ (1,0,1,0) \\ (1,0,1,0) \\ (1,0,1,0) \\ (1,1,1,0) \\ (1,1,1,1) \end{pmatrix}, \begin{pmatrix} (0,0,0,1) \\ (1,0,0,1) \\ (0,0,1,1) \\ (1,0,1,1) \\ (0,1,1,1) \\ (1,1,1,1) \end{pmatrix} \end{cases}$$

To construct  $(G/N_2)/(N_3/N_2)$ , we must first write down  $N_3/N_2$ .

$$N_3/N_2 = \left\{ \begin{bmatrix} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \end{bmatrix}, \begin{bmatrix} (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{bmatrix} \right\}$$

And since,

$$G/N_2 = \left\{ \begin{cases} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \end{cases}, \begin{cases} (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{cases}, \begin{cases} (0,0,0,1) \\ (1,0,0,1) \\ (0,1,0,1) \\ (1,1,0,1) \end{cases}, \begin{cases} (0,0,1,1) \\ (1,0,1,1) \\ (0,1,1,1) \\ (1,1,1,1) \end{cases} \right\},$$

we have that,

Again, notice the structural similarity between this and  $G/N_3$ .

$$G/N_3 = \begin{cases} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \\ (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{cases}, \begin{cases} (0,0,0,1) \\ (1,0,0,1) \\ (0,1,0,1) \\ (1,0,1,1) \\ (0,1,1,1) \\ (1,1,1,1) \end{cases}$$

And finally, we want to construct the cosets for  $[(G/N_1)/(N_2/N_1)]/[(N_3/N_1)/(N_2/N_1)]$ . We'll start first with  $N_2/N_1$ , then  $N_3/N_1$  followed by  $(N_3/N_1)/(N_2/N_1)$ .

$$\begin{split} N_2/N_1 = & \left\{ \begin{cases} (0,0,0,0) \\ (1,0,0,0) \end{cases}, \begin{cases} (0,1,0,0) \\ (1,1,0,0) \end{cases} \right\} \\ N_3/N_1 = & \left\{ \begin{cases} (0,0,0,0) \\ (1,0,0,0) \end{cases}, \begin{cases} (0,1,0,0) \\ (1,1,0,0) \end{cases}, \begin{cases} (0,0,1,0) \\ (1,0,1,0) \end{cases}, \begin{cases} (0,1,1,0) \\ (1,1,1,0) \end{cases} \right\} \\ & \left\{ (N_3/N_1)/(N_2/N_1) = \begin{cases} \begin{cases} (0,0,0,0) \\ (1,0,0,0) \\ (1,0,0,0) \end{cases}, \begin{cases} (0,1,0) \\ (1,0,1,0) \\ (1,1,1,0) \end{cases} \right\} \\ & \left\{ (0,1,1,0) \\ (1,1,1,0) \end{cases} \right\}. \end{split}$$

We previously found the following cosets for  $(G/N_1)/(N_2/N_1)$ .

$$(G/N_1)/(N_2/N_1) = \begin{cases} \left\{ \begin{cases} (0,0,0,0) \\ (1,0,0,0) \end{cases} \right\}, \\ \left\{ (0,1,0,0) \\ (1,1,0,0) \end{cases}, \\ \left\{ (0,1,1,0) \\ (1,1,0,0) \end{cases}, \\ \left\{ (0,1,1,0) \\ (1,1,0,0) \end{cases}, \\ \left\{ (0,1,0,1) \\ (1,1,0,1) \end{cases}, \\ \left\{ (0,1,0,1) \\ (1,1,0,1) \end{cases}, \\ \left\{ (0,1,1,1) \\ (1,1,1,1) \right\} \right\}$$

Hence, we can now write down the cosets for  $\lceil (G/N_1)/(N_2/N_1) \rceil / \lceil (N_3/N_1)/(N_2/N_1) \rceil$ .

Again, we want to notice the structural similarities between the three isomorphic groups.

$$G/N_3 = \begin{cases} \begin{pmatrix} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \\ (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{pmatrix}, \begin{pmatrix} (0,0,0,1) \\ (1,0,0,1) \\ (0,1,0,1) \\ (1,0,1,1) \\ (0,0,1,1) \\ (1,0,1,1) \\ (1,1,1,1) \end{pmatrix} \end{cases}$$

$$(G/N_2)/(N_3/N_2) = \begin{cases} \begin{cases} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \end{cases}, \begin{cases} (0,0,0,1) \\ (1,0,0,1) \\ (0,1,0,1) \\ (1,1,0,1) \end{cases} \\ \begin{cases} (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{cases}, \begin{cases} (0,0,0,1) \\ (0,1,0,1) \\ (1,0,1,1) \\ (0,1,1,1) \\ (1,1,1,1) \end{cases} \end{cases}$$

And finally, recall that each coset can be written as a representative element of that coset times the identity in our group of cosets. If we do this for each of our major constructions, then we arrive at the following. [NOTE: We will denote our operation as generic multiplication even though we are actually doing addition modulo 2 with regard to each coordinate.]

$$G/N_1 = \begin{cases} \{(0,0,0,0)\}, \{(0,1,0,0)\}, \{(0,0,1,0)\}, \{(0,0,0,1)\}, \{(0,0,0,1)\}, \{(1,0,0,0)\}, \{(1,0,0,0)\}, \{(1,0,0,0)\}, \{(1,0,0,0,0)\}, \{(1,0,0,0)\}, \{(1,0,0,0)\}, \{(1,0,0,0)\}, \{(1,0,0,0)\}, \{(1,0,0,0)\}, \{(1,0,0,0)\}, \{(0,$$

$$= \left\{ \begin{bmatrix} (0,0,0,0) \\ (1,1,0,0) \end{bmatrix}, (0,1,1,0) \\ (1,1,1,0) \end{bmatrix}, (0,1,0,1) \\ (0,1,0,1) \\ (1,1,1,0) \end{bmatrix}, (0,0,1,1) \end{bmatrix}, (0,0,0,0) \\ (0,1,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \end{bmatrix}, (0,0,1,0) \end{bmatrix}, (0,0,0,1) \\ (0,1,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \end{bmatrix}, (0,0,0,1) \end{bmatrix}, (0,0,0,1) \\ (0,1,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \end{bmatrix}, (0,0,1,1) \end{bmatrix}$$

$$\begin{aligned} & (G/N_1)/(N_2/N_1) = \begin{cases} \left\{ (0,0,0,0) \\ (1,0,0,0) \\ (1,1,0,0)$$

$$G/N_3 = \left\{ \begin{bmatrix} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (0,1,0,0) \\ (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{bmatrix}, \begin{bmatrix} (0,0,0,1) \\ (1,0,0,0) \\ (0,1,0,0) \\ (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,1) \\ (0,1,1,1) \\ (1,1,1,1) \end{bmatrix} \right\} = \left\{ \begin{bmatrix} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \\ (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{bmatrix}, (0,0,0,1) \right\} \left\{ \begin{bmatrix} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{bmatrix} \right\}$$

$$(G/N_2)/(N_3/N_2) = \begin{cases} \begin{cases} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \end{cases}, \begin{cases} (0,0,0,1) \\ (0,1,0,1) \\ (1,0,1,0) \\ (0,1,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{cases}, \begin{cases} (0,0,0,1) \\ (1,0,0,0) \\ (1,1,0,0) \end{cases}, \begin{cases} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \end{cases}, \begin{cases} (0,0,0,0) \\ (0,1,0,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{cases} \end{cases}$$

$$= \left\{ \begin{cases} \left\{ \begin{pmatrix} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \\ \end{pmatrix}, (0,0,0,1) \begin{cases} \left\{ \begin{pmatrix} (0,0,0,0) \\ (1,0,0,0) \\ (1,1,0,0) \\ (1,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \\ \end{pmatrix} \right\} \right\}$$

$$\begin{bmatrix} (G/N_1)/(N_2/N_1) \end{bmatrix} / \begin{bmatrix} (N_3/N_1)/(N_2/N_1) \end{bmatrix} = \begin{cases} \begin{cases} (0,0,0,0) \\ (1,0,0,0) \\ (1,1,0,0) \end{cases} \\ \begin{cases} (0,1,0,0) \\ (1,1,0,0) \end{cases} \\ \begin{cases} (0,0,1,0) \\ (1,0,1,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,1,0) \\ (1,0,1,0) \end{cases} \\ \begin{cases} (0,0,0,0) \\ (1,0,0,0) \end{cases} \\ \begin{cases} (0,0,0,0) \\ (1,0,0,0) \end{cases} \\ \begin{cases} (0,0,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,0,0) \\ (1,1,0,0) \end{cases} \\ \begin{cases} (0,0,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,0,0) \\ (1,0,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,0,0) \\ (1,0,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,0,0) \\ (1,0,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,0,0) \\ (1,1,0,0) \\ (1,1,1,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,0,0) \\ (1,1,0,0) \\ (1,1,1,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,0,0) \\ (1,1,0,0) \\ (1,1,1,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,0,0) \\ (1,0,0,0) \\ (1,1,0,0) \\ (1,1,1,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,0,0) \\ (1,1,0,0) \\ (1,1,1,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,0,0) \\ (1,1,0,0) \\ (1,1,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,1,0) \\ (1,1,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,1,0) \\ (1,1,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,1,0) \\ (1,1,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,1,0) \\ (1,1,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,1,0) \\ (1,1,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,1,0) \\ (1,1,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,1,0) \\ (1,1,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,1,0) \\ (1,1,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,1,0) \\ (1,1,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,1,0) \\ (1,1,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,1,0) \\ (1,1,0,0) \\ (1,1,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,1,0) \\ (1,1,0,0) \\ (1,1,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,1,0) \\ (1,1,0,0) \\ (1,1,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,1,0) \\ (1,1,0,0) \\ (1,1,0,0) \\ (1,1,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,1,0) \\ (1,1,0,0) \\ (1,1,0,0) \\ (1,1,0,0) \\ (1,1,0,0) \end{cases} \\ \end{cases} \\ \begin{cases} (0,0,1,0) \\ (1,1,0,0) \\ ($$

Among other things, this hopefully illustrates that as we continue to take quotients of quotients, every coset in the resulting quotient group can still be written as an element of G times the identity in that particular quotient of quotients.