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We are now going to examine not only some quotient groups, but also quotients of quotients and quotients of quotients of quotients in order to get a feel for what they are really like. Thus, let's start with the group $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $|G| = 16$, and the elements of G are,

$$G = \left\{ \begin{array}{l} (0,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (0,1,1,0), (0,1,0,1), (0,0,1,1), (0,1,1,1), \\ (1,0,0,0), (1,1,0,0), (1,0,1,0), (1,0,0,0), (1,1,1,0), (1,1,0,1), (1,0,1,1), (1,1,1,1) \end{array} \right\}$$

This group is abelian, and so all of its subgroups are normal. In particular, let's consider the following subgroups:

$$N_1 = \{(0,0,0,0), (1,0,0,0)\}$$

$$N_2 = \{(0,0,0,0), (1,0,0,0), (0,1,0,0), (1,1,0,0)\}$$

$$N_3 = \{(0,0,0,0), (1,0,0,0), (0,1,0,0), (1,1,0,0), (0,0,1,0), (1,0,1,0), (0,1,1,0), (1,1,1,0)\}$$

Notice that $N_1 \subset N_2 \subset N_3$. Also, $|G/N_1| = 8$, and the cosets in G/N_1 are,

$$G/N_1 = \left\{ \begin{array}{l} \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,1,0,0) \\ (1,1,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,1,0) \\ (1,0,1,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,0,1) \\ (1,0,0,1) \end{array} \right\}, \\ \left\{ \begin{array}{l} (0,1,1,0) \\ (1,1,1,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,1,0,1) \\ (1,1,0,1) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,1,1) \\ (1,0,1,1) \end{array} \right\}, \left\{ \begin{array}{l} (0,1,1,1) \\ (1,1,1,1) \end{array} \right\} \end{array} \right\}$$

Now we want to look at G/N_2 which by our isomorphism theorem is isomorphic to $(G/N_1)/(N_2/N_1)$. Thus, we'll first write down the cosets for G/N_2 and then compare this to the cosets in $(G/N_1)/(N_2/N_1)$. In particular, $|G/N_2| = 4$, and the cosets in G/N_2 are,

$$G/N_2 = \left\{ \begin{array}{l} \left[\begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \end{array} \right], \left[\begin{array}{l} (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{array} \right], \left[\begin{array}{l} (0,0,0,1) \\ (1,0,0,1) \\ (0,1,0,1) \\ (1,1,0,1) \end{array} \right], \left[\begin{array}{l} (0,0,1,1) \\ (1,0,1,1) \\ (0,1,1,1) \\ (1,1,1,1) \end{array} \right] \end{array} \right\}$$

Now we want to compare this to the cosets in $(G/N_1)/(N_2/N_1)$, so let's first write down N_2/N_1 .

$$N_2/N_1 = \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,1,0,0) \\ (1,1,0,0) \end{array} \right\} \right\}$$

Since,

$$G/N_1 = \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,1,0,0) \\ (1,1,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,1,0) \\ (1,0,1,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,0,1) \\ (1,0,0,1) \end{array} \right\}, \right. \\ \left. \left\{ \begin{array}{l} (0,1,1,0) \\ (1,1,1,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,1,0,1) \\ (1,1,0,1) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,1,1) \\ (1,0,1,1) \end{array} \right\}, \left\{ \begin{array}{l} (0,1,1,1) \\ (1,1,1,1) \end{array} \right\} \right\},$$

We have that,

$$(G/N_1)/(N_2/N_1) = \left\{ \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \end{array} \right\} \right\}, \left\{ \left\{ \begin{array}{l} (0,0,1,0) \\ (1,0,1,0) \end{array} \right\} \right\}, \left\{ \left\{ \begin{array}{l} (0,0,0,1) \\ (1,0,0,1) \end{array} \right\} \right\}, \left\{ \left\{ \begin{array}{l} (0,0,1,1) \\ (1,0,1,1) \end{array} \right\} \right\} \right\} \\ \left\{ \left\{ \left\{ \begin{array}{l} (0,1,0,0) \\ (1,1,0,0) \end{array} \right\} \right\}, \left\{ \left\{ \begin{array}{l} (0,1,1,0) \\ (1,1,0,0) \end{array} \right\} \right\}, \left\{ \left\{ \begin{array}{l} (0,1,0,1) \\ (1,1,0,1) \end{array} \right\} \right\}, \left\{ \left\{ \begin{array}{l} (0,1,1,1) \\ (1,1,1,1) \end{array} \right\} \right\} \right\}$$

Notice now the structural similarity between $(G/N_1)/(N_2/N_1)$ and G/N_2 .

$$G/N_2 = \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,0,1) \\ (1,0,0,1) \\ (0,1,0,1) \\ (1,1,0,1) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,1,1) \\ (1,0,1,1) \\ (0,1,1,1) \\ (1,1,1,1) \end{array} \right\} \right\}$$

And now we'll move on to G/N_3 which is isomorphic to both $(G/N_2)/(N_3/N_2)$ and $[(G/N_1)/(N_2/N_1)]/[(N_3/N_1)/(N_2/N_1)]$. As before, we'll start with the simplest quotient group, G/N_3 , and we'll methodically construct the other quotient groups so that we can observe the similarities. Thus, $|G/N_3| = 2$, and the cosets in G/N_3 are,

$$G/N_3 = \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \\ (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,0,1) \\ (1,0,0,1) \\ (0,1,0,1) \\ (1,1,0,1) \\ (0,0,1,1) \\ (1,0,1,1) \\ (0,1,1,1) \\ (1,1,1,1) \end{array} \right\} \right\}$$

To construct $(G/N_2)/(N_3/N_2)$, we must first write down N_3/N_2 .

$$N_3/N_2 = \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{array} \right\} \right\}$$

And since,

$$G/N_2 = \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,0,1) \\ (1,0,0,1) \\ (0,1,0,1) \\ (1,1,0,1) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,1,1) \\ (1,0,1,1) \\ (0,1,1,1) \\ (1,1,1,1) \end{array} \right\} \right\},$$

we have that,

$$(G/N_2)/(N_3/N_2) = \left\{ \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,0,1) \\ (1,0,0,1) \\ (0,1,0,1) \\ (1,1,0,1) \end{array} \right\} \right\}, \left\{ \left\{ \begin{array}{l} (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,1,1) \\ (1,0,1,1) \\ (0,1,1,1) \\ (1,1,1,1) \end{array} \right\} \right\} \right\}$$

Again, notice the structural similarity between this and G/N_3 .

$$G/N_3 = \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \\ (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,0,1) \\ (1,0,0,1) \\ (0,1,0,1) \\ (1,1,0,1) \\ (0,0,1,1) \\ (1,0,1,1) \\ (0,1,1,1) \\ (1,1,1,1) \end{array} \right\} \right\}$$

And finally, we want to construct the cosets for $[(G/N_1)/(N_2/N_1)]/[(N_3/N_1)/(N_2/N_1)]$.

We'll start first with N_2/N_1 , then N_3/N_1 followed by $(N_3/N_1)/(N_2/N_1)$.

$$N_2/N_1 = \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,1,0,0) \\ (1,1,0,0) \end{array} \right\} \right\}$$

$$N_3/N_1 = \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,1,0,0) \\ (1,1,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,1,0) \\ (1,0,1,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,1,1,0) \\ (1,1,1,0) \end{array} \right\} \right\}$$

$$(N_3/N_1)/(N_2/N_1) = \left\{ \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,1,0) \\ (1,0,1,0) \end{array} \right\} \right\}, \left\{ \left\{ \begin{array}{l} (0,1,0,0) \\ (1,1,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,1,1,0) \\ (1,1,1,0) \end{array} \right\} \right\} \right\}.$$

We previously found the following cosets for $(G/N_1)/(N_2/N_1)$.

$$(G/N_1)/(N_2/N_1) = \left\{ \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,1,0) \\ (1,0,1,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,0,1) \\ (1,0,0,1) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,1,1) \\ (1,0,1,1) \end{array} \right\} \right\}, \left\{ \left\{ \begin{array}{l} (0,1,0,0) \\ (1,1,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,1,1,0) \\ (1,1,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,1,0,1) \\ (1,1,0,1) \end{array} \right\}, \left\{ \begin{array}{l} (0,1,1,1) \\ (1,1,1,1) \end{array} \right\} \right\} \right\}$$

Hence, we can now write down the cosets for $[(G/N_1)/(N_2/N_1)]/[(N_3/N_1)/(N_2/N_1)]$.

$$[(G/N_1)/(N_2/N_1)]/[(N_3/N_1)/(N_2/N_1)] = \left\{ \left\{ \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,1,0) \\ (1,0,1,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,0,1) \\ (1,0,0,1) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,1,1) \\ (1,0,1,1) \end{array} \right\} \right\}, \left\{ \left\{ \begin{array}{l} (0,1,0,0) \\ (1,1,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,1,1,0) \\ (1,1,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,1,0,1) \\ (1,1,0,1) \end{array} \right\}, \left\{ \begin{array}{l} (0,1,1,1) \\ (1,1,1,1) \end{array} \right\} \right\} \right\} \right\}$$

Again, we want to notice the structural similarities between the three isomorphic groups.

$$G/N_3 = \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \\ (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,0,1) \\ (1,0,0,1) \\ (0,1,0,1) \\ (1,1,0,1) \\ (0,0,1,1) \\ (1,0,1,1) \\ (0,1,1,1) \\ (1,1,1,1) \end{array} \right\} \right\}$$

$$(G/N_2)/(N_3/N_2) = \left\{ \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,0,1) \\ (1,0,0,1) \\ (0,1,0,1) \\ (1,1,0,1) \end{array} \right\} \right\}, \left\{ \left\{ \begin{array}{l} (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,1,1) \\ (1,0,1,1) \\ (0,1,1,1) \\ (1,1,1,1) \end{array} \right\} \right\} \right\}$$

$$\left[(G/N_1)/(N_2/N_1) \right] / \left[(N_3/N_1)/(N_2/N_1) \right] = \left\{ \left\{ \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,0,1) \\ (1,0,0,1) \\ (0,1,0,1) \\ (1,1,0,1) \end{array} \right\} \right\}, \left\{ \left\{ \begin{array}{l} (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,1,1) \\ (1,0,1,1) \\ (0,1,1,1) \\ (1,1,1,1) \end{array} \right\} \right\} \right\} \right\}$$

And finally, recall that each coset can be written as a representative element of that coset times the identity in our group of cosets. If we do this for each of our major constructions, then we arrive at the following. [NOTE: We will denote our operation as generic multiplication even though we are actually doing addition modulo 2 with regard to each coordinate.]

$$G/N_3 = \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \\ (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,0,1) \\ (1,0,0,1) \\ (0,1,0,1) \\ (1,1,0,1) \\ (0,0,1,1) \\ (1,0,1,1) \\ (0,1,1,1) \\ (1,1,1,1) \end{array} \right\} \right\} = \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \\ (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{array} \right\}, (0,0,0,1) \right\}, \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \\ (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{array} \right\}$$

$$(G/N_2)/(N_3/N_2) = \left\{ \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \\ (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{array} \right\}, \left\{ \begin{array}{l} (0,0,0,1) \\ (1,0,0,1) \\ (0,1,0,1) \\ (1,1,0,1) \\ (0,0,1,1) \\ (1,0,1,1) \\ (0,1,1,1) \\ (1,1,1,1) \end{array} \right\} \right\}, \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \\ (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{array} \right\} \right\}, \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \\ (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{array} \right\}, (0,0,0,1) \right\}, \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \\ (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{array} \right\}$$

$$= \left\{ \left\{ \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \\ (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{array} \right\}, (0,0,0,1) \right\}, \left\{ \begin{array}{l} (0,0,0,0) \\ (1,0,0,0) \\ (0,1,0,0) \\ (1,1,0,0) \\ (0,0,1,0) \\ (1,0,1,0) \\ (0,1,1,0) \\ (1,1,1,0) \end{array} \right\} \right\}$$

$$\begin{aligned}
[(G/N_1)/(N_2/N_1)]/[(N_3/N_1)/(N_2/N_1)] &= \left\{ \left\{ \begin{array}{l} \{(0,0,0,0)\} \\ \{(1,0,0,0)\} \\ \{(0,1,0,0)\} \\ \{(1,1,0,0)\} \end{array} \right\}, \left\{ \begin{array}{l} \{(0,0,0,1)\} \\ \{(1,0,0,1)\} \\ \{(0,1,0,1)\} \\ \{(1,1,0,1)\} \end{array} \right\} \right\} \\
&\quad \left\{ \begin{array}{l} \{(0,0,1,0)\} \\ \{(1,0,1,0)\} \\ \{(0,1,1,0)\} \\ \{(1,1,1,0)\} \end{array} \right\}, \left\{ \begin{array}{l} \{(0,0,1,1)\} \\ \{(1,0,1,1)\} \\ \{(0,1,1,1)\} \\ \{(1,1,1,1)\} \end{array} \right\} \right\} \\
&= \left\{ \left\{ \begin{array}{l} \{(0,0,0,0)\} \\ \{(1,0,0,0)\} \\ \{(0,1,0,0)\} \\ \{(1,1,0,0)\} \end{array} \right\}, \left\{ \begin{array}{l} (0,0,0,1) \{(0,0,0,0)\} \\ (0,0,0,1) \{(1,0,0,0)\} \\ (0,0,0,1) \{(0,1,0,0)\} \\ (0,0,0,1) \{(1,1,0,0)\} \end{array} \right\} \right\} \\
&\quad \left\{ \begin{array}{l} \{(0,0,1,0)\} \\ \{(1,0,1,0)\} \\ \{(0,1,1,0)\} \\ \{(1,1,1,0)\} \end{array} \right\}, \left\{ \begin{array}{l} (0,0,0,1) \{(0,0,1,0)\} \\ (0,0,0,1) \{(1,0,1,0)\} \\ (0,0,0,1) \{(0,1,1,0)\} \\ (0,0,0,1) \{(1,1,1,0)\} \end{array} \right\} \right\} \\
&= \left\{ \left\{ \begin{array}{l} \{(0,0,0,0)\} \\ \{(1,0,0,0)\} \\ \{(0,1,0,0)\} \\ \{(1,1,0,0)\} \end{array} \right\}, \left\{ \begin{array}{l} \{(0,0,0,0)\} \\ \{(1,0,0,0)\} \\ \{(0,1,0,0)\} \\ \{(1,1,0,0)\} \end{array} \right\} \right\} \\
&\quad (0,0,0,1) \left\{ \begin{array}{l} \{(0,0,0,0)\} \\ \{(1,0,0,0)\} \\ \{(0,1,0,0)\} \\ \{(1,1,0,0)\} \end{array} \right\} \\
&\quad \left\{ \begin{array}{l} \{(0,0,1,0)\} \\ \{(1,0,1,0)\} \\ \{(0,1,1,0)\} \\ \{(1,1,1,0)\} \end{array} \right\}, \left\{ \begin{array}{l} \{(0,0,1,0)\} \\ \{(1,0,1,0)\} \\ \{(0,1,1,0)\} \\ \{(1,1,1,0)\} \end{array} \right\} \right\} \\
&\quad (0,0,0,1) \left\{ \begin{array}{l} \{(0,0,1,0)\} \\ \{(1,0,1,0)\} \\ \{(0,1,1,0)\} \\ \{(1,1,1,0)\} \end{array} \right\} \\
&= \left\{ \left\{ \begin{array}{l} \{(0,0,0,0)\} \\ \{(1,0,0,0)\} \\ \{(0,1,0,0)\} \\ \{(1,1,0,0)\} \end{array} \right\}, \left\{ \begin{array}{l} \{(0,0,0,0)\} \\ \{(1,0,0,0)\} \\ \{(0,1,0,0)\} \\ \{(1,1,0,0)\} \end{array} \right\} \right\} \\
&\quad (0,0,0,1) \left\{ \begin{array}{l} \{(0,0,0,0)\} \\ \{(1,0,0,0)\} \\ \{(0,1,0,0)\} \\ \{(1,1,0,0)\} \end{array} \right\} \\
&\quad \left\{ \begin{array}{l} \{(0,0,1,0)\} \\ \{(1,0,1,0)\} \\ \{(0,1,1,0)\} \\ \{(1,1,1,0)\} \end{array} \right\}, \left\{ \begin{array}{l} \{(0,0,1,0)\} \\ \{(1,0,1,0)\} \\ \{(0,1,1,0)\} \\ \{(1,1,1,0)\} \end{array} \right\} \right\} \\
&\quad (0,0,0,1) \left\{ \begin{array}{l} \{(0,0,1,0)\} \\ \{(1,0,1,0)\} \\ \{(0,1,1,0)\} \\ \{(1,1,1,0)\} \end{array} \right\}
\end{aligned}$$

Among other things, this hopefully illustrates that as we continue to take quotients of quotients, every coset in the resulting quotient group can still be written as an element of G times the identity in that particular quotient of quotients.