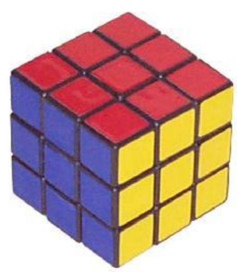
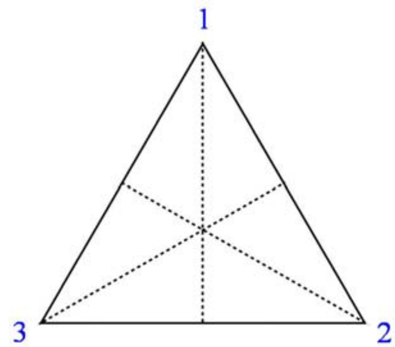
ORBITS, STABILIZERS, FIXERS, AND BURNSIDE'S COUNTING THEOREM

Consider this situation. You have just solved Rubik's cube, but you also instinctively know that if you rotated the cube $90°$ in any of six directions, then you would still consider the cube to still be in the same solved configuration. Thus, certain movements of the cube don't really result in what we consider a different configuration. And now our problem is this. Suppose we color the faces of the cube with six different colors and that we are also allowed to rotate the cube as described above. Then how many truly different color configurations are possible when we allow for rotations of the cube? This is the type of question we'll learn to answer in this chapter with the help of what we call orbits, stabilizers, fixers, and Burnside's Counting Theorem.



<u>Remember:</u> There are a few definitions and theorems you might want to recall before you wade any further into this section. First, when we say that *G* is a group that acts on a set of objects *X*, that means that each element of *G* corresponds to a permutation of the elements of *X*. For example, consider the equilateral triangle below with vertices labeled by 1, 2, or 3.



We can let $X = \{1,2,3\}$ and our set *G* can correspond to the permutations of these numbers created by either rotating the above triangle clockwise through angles that are multiples of $120°$ or by flipping the triangle about one of the indicated axes of symmetry or by some combination of these moves. By doing so, we can identify six distinct permutations which can be represented as follows.

$g_1 =$ the identity $= e = (\ )$
$g_2 = (1,2,3)$ [Recall that this permutation means $1 \to 2,\ 2 \to 3,\ \&\ 3 \to 1$]
$g_3 = (1,3,2)$
$g_4 = (2,3)$
$g_5 = (1,3)$
$g_6 = (1,2)$

We will use this example down below, so remember it. Also, recall that we denote the number of elements in a set or group by putting absolute value signs around the symbol for that set or group. Hence, for the group *G* above, we have $|G| = 6$.

And finally, recall that if *H* is a subgroup of a finite group *G*, then the <u>left coset</u> of *H* in *G* created by $a \in G$ is $aH = \{ah \mid h \in H\}$. Additionally, by Lagrange's Theorem, the number of left cosets of *H* in *G* is a divisor of *G* and is denoted by $[G:H] = \frac{|G|}{|H|}$. Notice that even though in the past we have generally examined right cosets, for the proofs that follow it will be easier this time to deal with left cosets.

<u>Definition:</u> Let *G* be a group that acts on a set *X*, and let $x \in X$. The <u>orbit of *x* by *G*</u> is the set $Orbit_G(x) = \{g(x) | g \in G\}$. In other words, the orbit of *x* consists of all elements of *X* that *x* can be changed into by the various elements of *G*.

<u>Theorem:</u> Let *G* be a group that acts on a set *X*, and let $\equiv$ be a relation on *X* defined by $x \equiv y$ if and only if $y = g(x)$ for some $g \in G$. Then $\equiv$ is an equivalence relation.

<u>Proof:</u> Recall that we need to show that this relationship is reflexive, symmetric, and transitive. Let's begin!

1. (reflexive) Let $e \in G$ be the identity element in *G*. Then, by definition, *e* leaves every element of *X* fixed so that $e(x) = x$. Hence, $x \equiv x$ and $\equiv$ is reflexive.

2. (symmetric) Suppose $x \equiv y$. Then there exists $g \in G$ such that $g(x) = y$. However, this implies that $g^{-1}(y) = x$ and that $y \equiv x$. Thus, $\equiv$ is symmetric.

3. (transitive) Suppose there exist $x, y, z \in X$ such that $x \equiv y$ and $y \equiv z$. Then there exist functions $g_1, g_2 \in G$ such that $g_1(x) = y$ and $g_2(y) = z$. Now let $g_3 = g_2 \circ g_1 \in G$. Then $g_3(x) = (g_2 \circ g_1)(x) = g_2(g_1(x)) = g_2(y) = z$. Therefore, $x \equiv z$ and $\equiv$ is transitive.

It now follows that $\equiv$ is an equivalence relation on *X*, and, hence, it partitions *X* into a series of disjoint subsets whose union is *X*. Also, it should be clear that each subset of this partition represents a single orbit created by the permutations in *G* when applied to the elements in the set *X*..

□

Corollary: If *x* and *y* belong to the same orbit, then $Orbit_G(x) = Orbit_G(y)$ and, consequently, $|Orbit_G(x)| = |Orbit_G(y)|$. (Recall that $|Orbit_G(x)|$ means the number of elements in $Orbit_G(x)$.)

Definition: Let *G* be a group that acts upon a set *X*, and let $x \in X$. Then the stabilizer of *x* by *G* is $Stabilizer_G(x) = G_x = \{g \in G \mid g(x) = x\}$.

Theorem: If *G* is a group that acts on a set *X*, and if $x \in X$, then the stabilizer of *x* by *G* is a subgroup of *G*.

Proof: To verify that $Stabilizer_G(x) = G_x$ is a subgroup of *G*, we need to show that for every $g \in Stabilizer_G(x) = G_x$ we have that $g^{-1} \in Stabilizer_G(x) = G_x$, and that for every $g_1, g_2 \in Stabilizer_G(x) = G_x$, we have that $g_1 \circ g_2 \in Stabilizer_G(x) = G_x$.

Thus, suppose $g \in Stabilizer_G(x) = G_x$. Then $x = e(x) = (g^{-1} \circ g)(x) = g^{-1}(g(x)) = g^{-1}(x)$. Hence, $g^{-1} \in Stabilizer_G(x) = G_x$.

Now suppose $g_1, g_2 \in Stabilizer_G(x) = G_x$. Then $(g_1 \circ g_2)(x) = g_1(g_2(x)) = g_1(x) = x$. Consequently, $g_1 \circ g_2 \in Stabilizer_G(x) = G_x$.

Therefore, it now follows that $Stabilizer_G(x) = G_x$ is a subgroup of *G*.

□

Theorem: If *G* is a finite group that acts on a set *X*, and if $x \in X$, then the number of elements in the orbit of *x* is $|Orbit_G(x)| = [G : G_x] = \frac{|G|}{|G_x|} = \frac{|G|}{|Stabilizer_G(x)|}$.

Proof: Since $Stabilizer_G(x) = G_x$ is a subgroup of *G*, we can consider the left cosets of G_x in *G*. In particular, notice that if $g_1, g_2 \in Stabilizer_G(x) = G_x$, then $g_1(x) = x = g_2(x)$. Now consider a left coset hG_x and suppose $h_1, h_2 \in hG_x$. Then $h_1 = hg_1$ & $h_2 = hg_2$ for some $g_1, g_2 \in G_x \Rightarrow h = h_1 g_1^{-1} \Rightarrow h_2 = hg_2 = (h_1 g_1^{-1}) g_2 = h_1(g_1^{-1} g_2) = h_1 g$ where $g = g_1^{-1} g_2 \in G_x$. Hence, $h_2(x) = (h_1 g)(x) = h_1(g(x)) = h_1(x)$. Thus, all elements in the same left coset of G_x yield the same value when applied to *x*.

Furthermore, if aG_x and bG_x are two different left cosets of G_x, then $a(x) \neq b(x)$ since, otherwise, if it were true that $a(x) = y = b(x)$, then

$(a^{-1}b)(x) = a^{-1}(b(x)) = a^{-1}(y) = x \Rightarrow a^{-1}b = g$ for some $g \in G_x \Rightarrow ag = a(a^{-1}b) = (aa^{-1})b$ $= e \cdot b = b \Rightarrow$ a and b belong to the same left coset of G_x. But this contradicts our assumption that $aG_x \neq bG_x$.

From the above it follows that we can find all the elements in the orbit of *x* by simply picking an arbitrary function from each left coset of G_x and applying it to *x*. In particular, the number of elements in the orbit of *x* is the same as the number of left cosets of G_x in *G*. Therefore, by Lagrange's Theorem, $\left|Orbit_G(x)\right| = \left[G : G_x\right] = \frac{|G|}{|G_x|} = \frac{|G|}{\left|Stabilizer_G(x)\right|}$.

□

Corollary: We can also rewrite $\left|Orbit_G(x)\right| = \frac{|G|}{|G_x|} = \frac{|G|}{\left|Stabilizer_G(x)\right|}$ as

$\left|G_x\right| = \left|Stabilizer_G(x)\right| = \frac{|G|}{\left|Orbit_G(x)\right|}$.

Definition: Let *G* be a group that acts upon a set *X*, and let $x \in X$. Then the fixer of *g* in *X* is $Fixer_X(g) = \left\{x \in X \,\middle|\, g(x) = x\right\}$.

Theorem: Let *G* be a group that acts on a set *X* and let $A = \left\{(g,x) \,\middle|\, g(x) = x \text{ where } g \in G \text{ and } x \in X\right\}$. Then the number of elements in *A*, denoted by $|A|$, is $|A| = \sum_{x \in X}\left|Stabilizer_G(x)\right| = \sum_{x \in X}\left|G_x\right| = \sum_{g \in G}\left|Fixer_X(g)\right|$.

Proof: The statement is obvious once you realize that $\sum_{x \in X}\left|Stabilizer_G(x)\right|$ and $\sum_{g \in G}\left|Fixer_X(g)\right|$ are just counting the same thing in two different ways. In $\sum_{x \in X}\left|Stabilizer_G(x)\right|$, we're fixing an $x \in X$ and then counting up all the functions $g \in G$ such that $g(x) = x$. And then we go on to the next $x \in X$. On the other hand, in $\sum_{g \in G}\left|Fixer_X(g)\right|$ we fix $g \in G$ and then count up the number of elements $x \in X$ such that $g(x) = x$. And then we move on to another $g \in G$.

As an example, suppose $X=\{1,2,3\}$, $G=\{g_1,g_2,g_3,g_4,g_5,g_6\}$, (as defined at the beginning of this chapter), and $A=\{(g_1,1),(g_1,2),(g_1,3),(g_4,1),(g_5,2),(g_6,3)\}$. Then $|A|=6$, and we can count this total in either of the two ways below.

x	**|Stabilizer(x)|**
1	2
2	2
3	2
	Sum=6

g	**|Fixer(g)|**
g1	3
g2	0
g3	0
g4	1
g5	1
g6	1
	Sum=6

In other words, 1 is stabilized by g_1 & g_4, 2 is stabilized by g_1 & g_5, and 3 is stabilized by g_1 & g_6. On the other hand, g_1 fixes 1, 2, & 3, g_2 and g_3 fix no elements in *X*, g_4 fixes 1, g_5 fixes 2, and g_6 fixes 3. Either way, the sum is the same. Thus,
$|A|=\sum_{x\in X}|Stabilizer_G(x)|=\sum_{x\in X}|G_x|=\sum_{g\in G}|Fixer_X(g)|$.

□

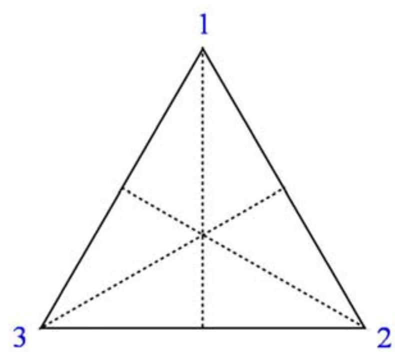
<u>Burnside's Counting Theorem:</u> If *G* is a finite group that acts on a set *X*, then the number of orbits created by *G* acting on *X* is $\frac{1}{|G|}\sum_{x\in X}|G_x|=\frac{1}{|G|}\sum_{x\in X}|Stabilizer_G(x)|=\frac{1}{|G|}\sum_{g\in G}|Fixer_X(g)|$.

<u>Proof:</u> At this point, we have pretty much developed all the pieces of the puzzle, and we just need to put them together. Recall that our corollary above says that
$|G_x|=|Stabilizer_G(x)|=\frac{|G|}{|Orbit_G(x)|}$. Hence,
$\frac{1}{|G|}\sum_{x\in X}|G_x|=\frac{1}{|G|}\sum_{x\in X}\frac{|G|}{|Orbit_G(x)|}=\frac{|G|}{|G|}\sum_{x\in X}\frac{1}{|Orbit_G(x)|}=\sum_{x\in X}\frac{1}{|Orbit_G(x)|}$. Now what is this last expression going to add up to? Well, suppose, for example, that one particular orbit by a group *G* contains just three points – a, b, and c. In this case,
$Orbit_G(a)=Orbit_G(b)=Orbit_G(c)$, and $|Orbit_G(a)|=|Orbit_G(b)|=|Orbit_G(c)|=3$.
Consequently, $\frac{1}{|Orbit_G(a)|}+\frac{1}{|Orbit_G(b)|}+\frac{1}{|Orbit_G(c)|}=\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1$. Similarly, if an orbit produced by a group *G* consisted of four elements, d, e, f, and g, then we would have
$\frac{1}{|Orbit_G(d)|}+\frac{1}{|Orbit_G(e)|}+\frac{1}{|Orbit_G(f)|}+\frac{1}{|Orbit_G(g)|}=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=1$. Thus, if we arrange the sum $\sum_{x\in X}\frac{1}{|Orbit_G(x)|}$ in such a way that we add up all the terms corresponding to the elements of one orbit before going on to the next orbit, then the sum simply becomes $1+1+\ldots+1$ where the term "1" occurs as many times as there are distinct orbits in *X*

produced by the action of the group G. In other words,
$\frac{1}{|G|}\sum_{x\in X}|G_x| = \frac{1}{|G|}\sum_{x\in X}\frac{|G|}{|Orbit_G(x)|} = \frac{|G|}{|G|}\sum_{x\in X}\frac{1}{|Orbit_G(x)|} = \sum_{x\in X}\frac{1}{|Orbit_G(x)|}$ is equal to that number of orbits produced on X by G. And since one of our theorems above demonstrated that $\sum_{x\in X}|Stabilizer_G(x)| = \sum_{x\in X}|G_x| = \sum_{g\in G}|Fixer_G(g)|$, we can also write this result as
$\frac{1}{|G|}\sum_{x\in X}|Stabilzer_G(x)| = \frac{1}{|G|}\sum_{g\in G}|Fixer_X(g)|$ is equal to the number of orbits on X produced by G.

□

Example 1: Let's apply this theorem to the example at the top of this chapter where G is the group of six permutations we can make of the elements of $X = \{1,2,3\}$.



$g_1 =$ the identity $= e = (\ )$
$g_2 = (1,2,3)$
$g_3 = (1,3,2)$
$g_4 = (2,3)$
$g_5 = (1,3)$
$g_6 = (1,2)$

On the one hand, it should be clear that there is only one orbit consisting of $\{1,2,3\}$. This is true because we can change each of these elements into any of the others just by repeated applications of a clockwise rotation of our triangle.

x	|Stabilizer(x)|
1	2
2	2
3	2
	Sum=6

g	|Fixer(g)|
g1	3
g2	0
g3	0
g4	1
g5	1
g6	1
	Sum=6

Additionally, by counting up for each $x \in X$ the number of elements in $Stabilizer(x)$, and by counting up for each $g \in G$ the number of elements in $Fixer(g)$, we obtain the same result from Burnside's Counting Theorem,

$$\frac{1}{|G|}\sum_{x \in X}|Stabilzer_G(x)| = \frac{1}{|G|}\sum_{g \in G}|Fixer_X(g)| = \frac{1}{6}\cdot 6 = 1$$

Example 2: Let $X = \{1,2,3,4\}$ and let $G = \{(\ ),(1,2),(3,4),(1,2)(3,4)\}$, $|G| = 4$. This group is called the *Klein 4*-group, and it is analogous to the states that can result when you have two lamps, one to your left and one to your right. You can leave both lamps off (the identity), or you can turn on the lamp on your left, or you can turn on the lamp on your right, or you can turn on both lamps. Each transposition in our group *G*, $(1,2)$ and $(3,4)$, is analogous to flipping a switch on a lamp on or off.

Now as for the number of orbits that *X* will have under the action of *G*, it should be clear that there are two. We can change 1 to 2 and we can change 3 to 4 and that's it. Thus, we might write $Orbit1 = \{1,2\}$ and $Orbit2 = \{3,4\}$. And if we count the orbits using Burnside's Counting Theorem, then once again we get the same thing.

x	|Stabilizer(x)|
1	2
2	2
3	2
4	2
	Sum=8

g	|Fixer(g)|
( )	4
(1,2)	2
(3,4)	2
(1,2)(3,4)	0
	Sum=8

Hence, $\frac{1}{|G|}\sum_{x \in X}|Stabilzer_G(x)| = \frac{1}{4}\cdot 8 = 2$ and $\frac{1}{|G|}\sum_{g \in G}|Fixer_X(g)| = \frac{1}{4}\cdot 8 = 2$.

Example 3: Let $X = \{1,2,3\}$ and let $G = \{(\ ),(1,2,3),(1,3,2)\}$, $|G| = 3$. Again, since the permutations in *G* can change 1 into 2 and 1 into 3, there should be only one orbit, $Orbit1 = \{1,2,3\}$. We can confirm this using Burnside's Counting Theorem.

x	|Stabilizer(x)|
1	1
2	1
3	1
	Sum=3

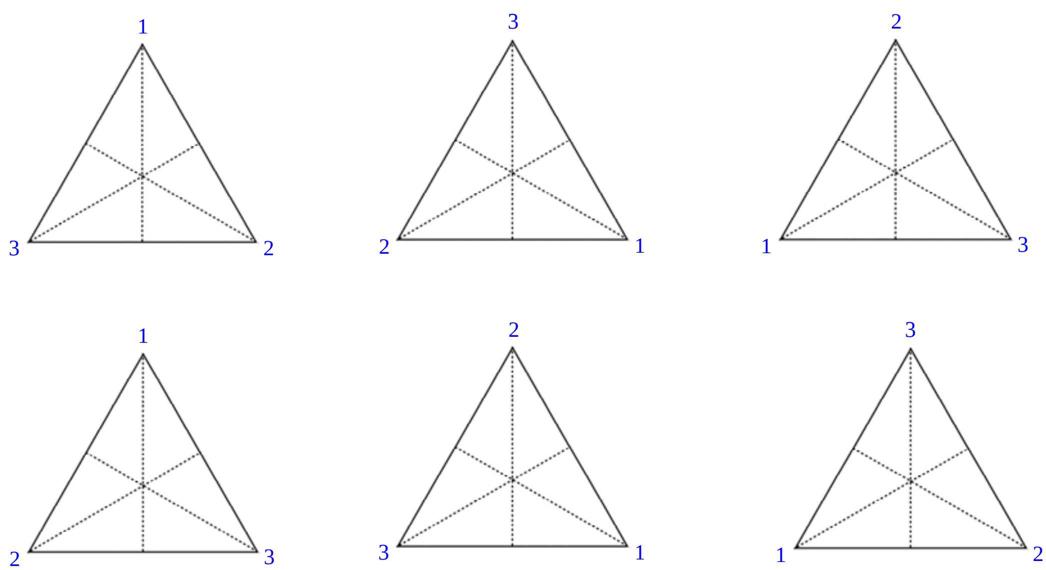
g	|Fixer(g)|
( )	3
(1,2,3)	0
(1,3,2)	0
	Sum=3

Thus, the number of orbits is $\frac{1}{|G|}\sum_{x\in X}|Stabilzer_G(x)| = \frac{1}{|G|}\sum_{g\in G}|Fixer_X(g)| = \frac{1}{3}\cdot 3 = 1$.

Notice, too, that if we label the vertices of an equilateral triangle with the number 1, 2, and 3, then we can interpret the permutations in *G* as corresponding to clockwise rotations of $0°$, $120°$, and $240°$, respectively.

Example 4: Let *X* equal the set of all distinct arrangements of the numbers 1, 2, and 3 on the vertices of an equilateral triangle, and let $G = \{(\ ),(1,2,3),(1,3,2)\}$, $|G| = 3$.

Notice that the permutations in our group *G* can once again be thought of as clockwise rotations of our triangle through angles that are multiples of $120°$, but our set *X* is different from what it was in the previous example. In particular, *X* is comprised of the following six arrangements:



Using Burnside's Counting Theorem, we discover that there are two orbits.

x	|Stabilizer(x)|
1	1
2	1
3	1
4	1
5	1
6	1
	Sum=6

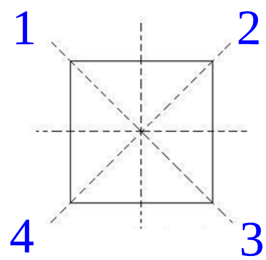
g	|Fixer(g)|
( )	6
(1,2,3)	0
(1,3,2)	0
	Sum=6

The number of orbits is $\frac{1}{|G|}\sum_{x\in X}|Stabilzer_G(x)| = \frac{1}{|G|}\sum_{g\in G}|Fixer_X(g)| = \frac{1}{3}\cdot 6 = 2$.

Notice that $Orbit1$ could be the configurations of the triangles in the first row above, and $Orbit2$ corresponds to the configurations in the second row above.

Example 5: Suppose you have four colors, red, green, blue, and yellow, and you paint each edge of a square a different color, and let X be the set of all possible color configurations. For example, on such configuration cold be top=red, bottom=blue, left=green, and right=yellow, and another possible configuration would be top=green, bottom=red, left=blue, and right=green. In all, the number of possible configurations is $4! = 4\cdot 3\cdot 2\cdot 1 = 24$. This is because we have four choices for the top color, then three left for the bottom color, two choices for the left side color, and then only one choice left for the right side color.

For our group, we will use D_4, the symmetries of a square. In other words, we can rotate our square clockwise through angles that are multiples of $90°$, or we can flip our square about any of four axes of symmetry.



The number of elements in this group is eight, $|D_4| = 8$, and if we label the vertices of our square 1, 2, 3, and 4, then we can represent D_4 as the following set of permutations, $D_4 = \{(\ ),(1,2,3,4),(1,4,3,2),(2,4),(1,3),(1,2)(3,4),(1,3)(2,4),(1,4)(2,3)\}$.

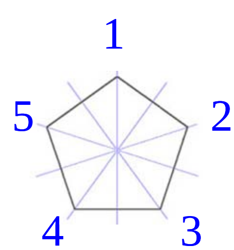
If two color configurations are in the same orbit, then we can change one into the other through some sequence of rotations and flips. Thus, the number of truly distinct color configurations possible is equal to the number of orbits in X created by D_4. Fortunately, this is easy to count. All we need to realize is that the identity keeps all 24 color configurations fixed while every other rotation or flip keeps none of the color configurations fixed.

g	|Fixer(g)|
( )	24
(1,2,3,4)	0
(1,4,3,2)	0
(2,4)	0
(1,3)	0
(1,2)(3,4)	0
(1,3)(2,4)	0
(1,4)(2,3)	0
	Sum=24

Hence, the number of orbits is $\frac{1}{|D_4|}\sum_{x\in X}|Stabilzer_{D_4}(x)| = \frac{1}{|D_4|}\sum_{g\in D_4}|Fixer_X(g)| = \frac{1}{8}\cdot 24 = 3$.

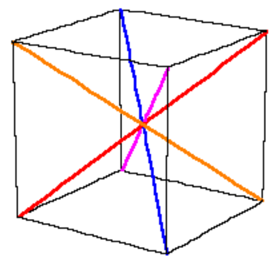
Example 6: Suppose you have a pentagonal bracelet with 5 differently colored, equally spaced beads, and suppose that you either rotate the bracelet clockwise through multiples of $72°$, or you can flip the bracelet about any of 5 axes of symmetry. Then our set X will consist of $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$ color configurations, and our group is D_5, the group of symmetries of a regular pentagon with $|D_5| = 10$. Again, if we label the vertices 1, 2, 3, 4, and 5, then we can describe D_5 in terms of the following permutations,

$$D_5 = \left\{ \begin{matrix} (\ ),(1,2,3,4,5),(1,3,5,2,4),(1,4,2,5,3),(1,5,4,3,2), \\ (2,5)(3,4),(1,2)(3,5),(1,3)(4,5),(1,4)(2,3),(1,5)(2,4) \end{matrix} \right\}.$$



Now, as before, the identity fixes all $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$ color configurations, and the remaining elements of D_5 fix none of the configurations. Hence, the number of orbits in X under D_5 is $\frac{1}{|D_5|}\sum_{x\in X}|Stabilzer_{D_5}(x)| = \frac{1}{|D_5|}\sum_{g\in D_5}|Fixer_X(g)| = \frac{1}{10}\cdot 120 = 12$.

Example 7: We'll now give a quick answer to the problem we posed at the beginning of this chapter where we can paint the six faces of a cube with six colors such that each color is used only once. This allows for $6! = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 720$ ways to paint the cube. However, we also allow rotations that are multiples of $90°$ in any of six directions, and this will make some of our coloring schemes equivalent to others. In particular, two color schemes will be equivalent if the are in the same orbit created by our rotation group *G*, and so the number of distinct color schemes will be the same as the number of orbits that *G* creates when it acts on our colored cube. Using the same logic as before, we can say that the identity of our group fixes all 720 coloring schemes, but every other element in *G* changes one color scheme into another. Thus, the total number of coloring schemes fixed by *G* is 720. Now, however, we need to know how many elements there are in *G*, but that's not hard to count if we realize that our cube has four diagonal lines going through the center, and our rotations can create any possible permutation of these four diagonal lines.



Hence, $|G| = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, and from this it follows that the number of orbits created by *G* for the coloring schemes for our cube is
$\frac{1}{|G|}\sum_{x \in X} |Stabilzer_G(x)| = \frac{1}{|G|}\sum_{g \in G} |Fixer_X(g)| = \frac{1}{24} \cdot 720 = 30$. In other words, there are 30 ways to color the faces of our cube that are distinct from one another when we allow for rotations of the cube.