ORBITS, STABILIZERS, FIXERS, AND BURNSIDE'S COUNTING THEOREM – EXERCISES

1. Let $X = \{1,2,3,4\}$ and let $G = D_4 = \{(\ ), (1,2,3,4), (1,4,3,2), (2,4), (1,3), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$. Use Burnside's Counting Theorem to find the number of orbits on *X* created by D_4.

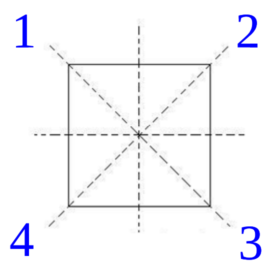
2. Let $X = \{1,2,3,4,5,6\}$ and let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(\ ), (5,6), (3,4), (3,4)(5,6), (1,2), (1,2)(5,6), (1,2)(3,4), (1,2)(3,4)(5,6)\}$. Use Burnside's Counting Theorem to find the number of orbits on *X* created by G.

3. Let $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ and let
$$G = \mathbb{Z}_3 \rtimes \mathbb{Z}_4 = \left\{ \begin{array}{l} (\ ), (1,2,3,9)(4,12,7,6)(5,11,8,10), (1,3)(2,9)(4,7)(5,8)(6,12)(10,11), \\ (1,4,8)(2,10,12)(3,7,5)(6,9,11), (1,5,4,3,8,7)(2,6,10,9,12,11), \\ (1,6,3,12)(2,8,9,5)(4,11,7,10), (1,7,8,3,4,5)(2,11,12,9,10,6), \\ (1,8,4)(2,12,10)(3,5,7)(6,11,9), (1,9,3,2)(4,6,7,12)(5,10,8,11), \\ (1,10,3,11)(2,7,9,4)(5,6,8,12), (1,11,3,10)(2,4,9,7)(5,12,8,6), \\ (1,12,3,6)(2,5,9,8)(4,10,7,11) \end{array} \right\}$$
Use Burnside's Counting Theorem to find the number of orbits on *X* created by G.

4. Let $X = \{1,2,3,4,5,6,7,8\}$ and let
$$G = Q = \left\{ \begin{array}{l} (\ ), (1,2,5,6)(3,8,7,4), (1,3,5,7)(2,4,6,8), (1,4,5,8)(2,7,6,3), \\ (1,5)(2,6)(3,7)(4,8), (1,6,5,2)(3,4,7,8), (1,7,5,3)(2,8,6,4), (1,8,5,4)(2,3,6,7) \end{array} \right\}.$$
Use Burnside's Counting Theorem to find the number of orbits on *X* created by G.

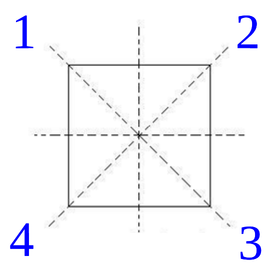
5. Suppose you have four colors, red, green, blue, and yellow, and you paint each edge of a square a different color, and let *X* be the set of all possible color configurations. For example, on such configuration cold be top=red, bottom=blue, left=green, and right=yellow, and another possible configuration would be top=green, bottom=red, left=blue, and right=green. In all, the number of possible configurations is $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. This is because we have four choices for the top color, then three left for the bottom color, two choices for the left side color, and then only one choice left for the right side color.

   For our group, we will use C_4, the rotations of a square. In other words, we can rotate our square clockwise through angles that are multiples of $90°$.



Thus, we can represent our group as $C_4 = \{(\ ), (1,2,3,4), (1,3)(2,4), (1,4,3,2)\}$. Use Burnside's Counting Theorem to find the number of orbits on *X* created by C_4.

6. Suppose you have a bracelet with 4 differently colored, equally spaced beads, and suppose that you either rotate the bracelet clockwise through multiples of $90°$, or you can flip the bracelet about any of 4 axes of symmetry. Then our set *X* will consist of $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ color configurations, and our group is D_4, the group of symmetries of a square with $|D_4| = 8$. Again, if we label the vertices 1, 2, 3, and 4, then we can describe D_4 in terms of the following permutations, $D_4 = \{(\ ), (1,2,3,4), (1,4,3,2), (2,4), (1,3), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$.



Use Burnside's Counting Theorem to find the number of orbits on *X* created by D_4.

7. The number of permutations that can be made from *n* objects if we choose *r* is ${}_nP_r = (n)(n-1)\ldots(n-r+1)$. For example, ${}_4P_3 = (4)(3)(2) = 24$. Let *X* be the set of all 24 permutations of the letters a, b, c, and d when we select just three letters, and let *G* be the set of all permutations of three objects. Use Burnside's Counting Theorem to show that the number of combinations that can be made from four objects when we choose three is ${}_4C_3 = \frac{{}_4P_3}{3!}$. Conclude that, in general, ${}_nC_r = \frac{{}_nP_r}{r!}$