ORBITS, STABILIZERS, FIXERS, AND BURNSIDE'S COUNTING THEOREM – EXERCISES – ANSWERS

1. Let $X = \{1,2,3,4\}$ and let $G = D_4 = \{(\ ),(1,2,3,4),(1,4,3,2),(2,4),(1,3),(1,2)(3,4),(1,3)(2,4),(1,4)(2,3)\}$. Use Burnside's Counting Theorem to find the number of orbits on *X* created by D_4.

   We'll count both the number of stabilizers and the number of fixers, and we'll verify that the sums are the same.

x	|Stabilizer(x)|
1	2
2	2
3	2
4	2
	Sum=8

g	|Fixer(g)|
( )	4
(1,2,3,4)	0
(1,4,3,2)	0
(2,4)	2
(1,3)	2
(1,2)(3,4)	0
(1,3)(2,4)	0
(1,4)(2,3)	0
	Sum=8

   Therefore, the number of orbits on *X* created by D_4 is

$$\frac{1}{|D_4|}\sum_{x\in X}\left|Stabilzer_{D_4}(x)\right| = \frac{1}{|D_4|}\sum_{g\in D_4}\left|Fixer_X(g)\right| = \frac{1}{8}\cdot 8 = 1.$$

2. Let $X = \{1,2,3,4,5,6\}$ and let $G = \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2 = \{(\ ),\ (5,6),\ (3,4),\ (3,4)(5,6),\ (1,2),\ (1,2)(5,6),\ (1,2)(3,4),\ (1,2)(3,4)(5,6)\}$. Use Burnside's Counting Theorem to find the number of orbits on *X* created by G.

   We'll count both the number of stabilizers and the number of fixers, and we'll verify that the sums are the same.

x	|Stabilizer(x)|
1	4
2	4
3	4
4	4
5	4
6	4
	Sum=24

g	|Fixer(g)|
( )	6
(5,6)	4
(3,4)	4
(3,4)(5,6)	2
(1,2)	4
(1,2)(5,6)	2
(1,2)(3,4))	2
(1,2)(3,4)(5,6)	0
	Sum=24

   Therefore, the number of orbits on *X* created by *G* is

$$\frac{1}{|G|}\sum_{x\in X}|Stabilzer_G(x)| = \frac{1}{|G|}\sum_{g\in G}|Fixer_X(g)| = \frac{1}{8}\cdot 24 = 3.$$

Also, a little inspection reveals that the three orbits are:

$Orbit1 = \{1,2\}$,
$Orbit2 = \{3,4\}$,
$Orbit3 = \{5,6\}$.

3. Let $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ and let

$$G = \mathbb{Z}_3 >\!\!\lhd \mathbb{Z}_4 = \left\{\begin{array}{l} (\ ),\ (1,2,3,9)(4,12,7,6)(5,11,8,10),\ (1,3)(2,9)(4,7)(5,8)(6,12)(10,11), \\ (1,4,8)(2,10,12)(3,7,5)(6,9,11),\ (1,5,4,3,8,7)(2,6,10,9,12,11), \\ (1,6,3,12)(2,8,9,5)(4,11,7,10),\ (1,7,8,3,4,5)(2,11,12,9,10,6), \\ (1,8,4)(2,12,10)(3,5,7)(6,11,9),\ (1,9,3,2)(4,6,7,12)(5,10,8,11), \\ (1,10,3,11)(2,7,9,4)(5,6,8,12),\ (1,11,3,10)(2,4,9,7)(5,12,8,6), \\ (1,12,3,6)(2,5,9,8)(4,10,7,11) \end{array}\right\}$$

Use Burnside's Counting Theorem to find the number of orbits on *X* created by G.

If we think about this, it becomes obvious that, except for the identity, each permutation in *G* moves all twelve elements in *X*. Hence, for each $x \in X$ we have that $|Stabilzer_G(x)| = 1$, and, hence, $\sum_{x\in X}|Stabilzer_G(x)| = 12$. Similarly, for each $g \in G$, the only permutation that fixes anything in *X* is the identity, and the identity fixes all twelve elements in *X*. Thus, $\sum_{g\in G}|Fixer_X(g)| = 12$. And finally, since $|G| = 12$, the number of orbits on *X* created by *G* is

$$\frac{1}{|G|}\sum_{x\in X}|Stabilzer_G(x)| = \frac{1}{|G|}\sum_{g\in G}|Fixer_X(g)| = \frac{1}{12}\cdot 12 = 1.$$

4. Let $X = \{1,2,3,4,5,6,7,8\}$ and let

$$G = Q = \left\{\begin{array}{l} (\ ),(1,2,5,6)(3,8,7,4),(1,3,5,7)(2,4,6,8),(1,4,5,8)(2,7,6,3), \\ (1,5)(2,6)(3,7)(4,8),(1,6,5,2)(3,4,7,8),(1,7,5,3)(2,8,6,4),(1,8,5,4)(2,3,6,7) \end{array}\right\}.$$

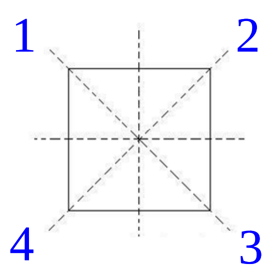
Use Burnside's Counting Theorem to find the number of orbits on *X* created by G.

The group used here is the Quaternion group which is thought of as both a generalization of vectors and complex numbers. The number of elements in *G* is $|G| = 8$. Furthermore, *X* has eight elements, and the only permutation in *G* that fixes any of these elements is the identity. Thus, $\sum_{x\in X}|Stabilzer_G(x)| = 8 = \sum_{g\in G}|Fixer_X(g)|$, and

the number of orbits on *X* created by *G* is
$\frac{1}{|G|}\sum_{x \in X} |Stabilzer_G(x)| = \frac{1}{|G|}\sum_{g \in G} |Fixer_X(g)| = \frac{1}{8}\cdot 8 = 1$.

5. Suppose you have four colors, red, green, blue, and yellow, and you paint each edge of a square a different color, and let *X* be the set of all possible color configurations. For example, on such configuration cold be top=red, bottom=blue, left=green, and right=yellow, and another possible configuration would be top=green, bottom=red, left=blue, and right=green. In all, the number of possible configurations is $4! = 4\cdot 3\cdot 2\cdot 1 = 24$. This is because we have four choices for the top color, then three left for the bottom color, two choices for the left side color, and then only one choice left for the right side color.

   For our group, we will use C_4, the rotations of a square. In other words, we can rotate our square clockwise through angles that are multiples of $90°$.



   Thus, we can represent our group as $C_4 = \{(\ ), (1,2,3,4), (1,3)(2,4), (1,4,3,2)\}$. Use Burnside's Counting Theorem to find the number of orbits on *X* created by C_4.

   Clearly, $|C_4| = 4$ and $|X| = 24$. Additionally, the only permutation in C_4 that fixes anything in *X* is the identity.

g	|Fixer(g)|
( )	24
(1,2,3,4)	0
(1,3)(2,4)	0
(1,4,3,2)	0
	Sum=24

   Therefore, the number of orbits on *X* created by C_4 is $\frac{1}{|C_4|}\sum_{g \in C_4} |Fixer_X(g)| = \frac{1}{4}\cdot 24 = 6$.

6. Suppose you have a bracelet with 4 differently colored, equally spaced beads, and suppose that you either rotate the bracelet clockwise through multiples of $90°$, or you can flip the bracelet about any of 4 axes of symmetry. Then our set *X* will consist of

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ color configurations, and our group is D_4, the group of symmetries of a square with $|D_4| = 8$. Again, if we label the vertices 1, 2, 3, and 4, then we can describe D_4 in terms of the following permutations,
$D_4 = \{(\ ), (1,2,3,4), (1,4,3,2), (2,4), (1,3), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$.

Use Burnside's Counting Theorem to find the number of orbits on *X* created by D_4.

As we've seen before, every element in D_4 moves all 24 elements in *X* except for the identity which fixes those 24 elements. Hence, the number of orbits on *X* created by D_4 is $\frac{1}{|D_4|}\sum_{g \in D_4} |Fixer_X(g)| = \frac{1}{8} \cdot 24 = 3$.

7. The number of permutations that can be made from *n* objects if we choose *r* is ${}_nP_r = (n)(n-1)\ldots(n-r+1)$. For example, ${}_4P_3 = (4)(3)(2) = 24$. Let *X* be the set of all 24 permutations of the letters a, b, c, and d when we select just three letters, and let *G* be the set of all permutations of three objects. Use Burnside's Counting Theorem to show that the number of combinations that can be made from four objects when we choose three is ${}_4C_3 = \frac{{}_4P_3}{3!}$. Conclude that, in general, ${}_nC_r = \frac{{}_nP_r}{r!}$

Consider the arrangement $abc \in X$ and the permutation $(1,2,3) \in G$. Then by $(1,2,3)[abc]$ we mean move the 1st letter to the 2nd position, move the 2nd letter to the 3rd position, and move the 3rd letter to the 1st position. In other words, $(1,2,3)[abc] = cab$. Also, the complete set of permutations in *G* is $G = \{(\ ), (1,2,3), (1,3,2), (1,2), (1,3), (2,3)\}$. Notice, too, that the only permutation in *G* that fixes any element in *X* is the identity and $|Fixer_X(identity)| = 24$. Therefore, the number of orbits on *X* created by *G* is $\frac{1}{|G|}\sum_{g \in G} |Fixer_X(g)| = \frac{1}{6} \cdot 24 = \frac{{}_4P_3}{3!} = 4$. In particular, the four orbits are:

$Orbit1 = \{abc, acb, bac, bca, cab, cba\}$,
$Orbit2 = \{abd, adb, bad, bda, dab, dba\}$,
$Orbit3 = \{acd, adc, cad, cda, dac, dca\}$,
$Orbit4 = \{bcd, bdc, cbd, cdb, dbc, dcb\}$.

Notice that each orbit consists of all six permutations of three given letters. Additionally, this argument can be easily generalized to show that ${}_nC_r = \frac{{}_nP_r}{r!}$.