

TRIPLE INTEGRALS - ANSWERS

For each of the following problems write a triple integral and evaluate it.

1. Find the volume of the solid in the first octant bounded above by the plane $x + y + z = 1$ and below by the xy -plane.

$$\begin{aligned} \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz dy dx &= \int_0^1 \int_0^{1-x} (1-x-y) dy dx = \int_0^1 \left(y - xy - \frac{y^2}{2} \right) \Big|_0^{1-x} dx \\ &= \int_0^1 \left(\frac{x^2}{2} - x + \frac{1}{2} \right) dx = \frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{2} \Big|_0^1 = \frac{1}{6} - \frac{1}{2} + \frac{1}{2} = \frac{1}{6} \end{aligned}$$

2. Find the volume of the solid between $z = x^2 + y^2 + 1$ and $z = -x^2 - y^2 - 1$ where $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.

$$\begin{aligned} \int_{-1}^1 \int_{-1-x^2-y^2-1}^{x^2+y^2+1} dz dy dx &= \int_{-1}^1 \int_{-1}^1 (x^2 + y^2 + 1) - (-x^2 - y^2 - 1) dy dx \\ &= \int_{-1}^1 \int_{-1}^1 (2x^2 + 2y^2 + 2) dy dx = \int_{-1}^1 \left(2x^2 y + \frac{2y^3}{3} + 2y \right) \Big|_{-1}^1 dx \\ &= \int_{-1}^1 \left(4x^2 + \frac{4}{3} + 4 \right) dx = \left(\frac{4x^3}{3} + \frac{4x}{3} + 4x \right) \Big|_{-1}^1 = \left(\frac{4}{3} + \frac{4}{3} + 4 \right) - \left(-\frac{4}{3} - \frac{4}{3} - 4 \right) \\ &= \frac{8}{3} + \frac{8}{3} + \frac{24}{3} = \frac{40}{3} \end{aligned}$$

3. Evaluate $\iiint_V x e^{-xy} dV$ where V is the solid bounded by the planes $x = -1$, $x = 1$, $y = 0$, $y = 2$, $z = 0$, and $z = 3$.

$$\begin{aligned} \int_{-1}^1 \int_0^2 \int_0^3 x e^{-xy} dz dy dx &= \int_{-1}^1 \int_0^2 x e^{-xy} z \Big|_0^3 dy dx = \int_{-1}^1 \int_0^2 3x e^{-xy} dy dx = \int_{-1}^1 3e^{-xy} \Big|_0^2 dx \\ &= \int_{-1}^1 (3e^{2x} - 3) dx = \left(\frac{3e^{2x}}{2} - 3x \right) \Big|_{-1}^1 = \left(\frac{3e^2}{2} - 3 \right) - \left(\frac{3e^{-2}}{2} + 3 \right) = \frac{3(e^2 - e^{-2})}{2} - 6 \end{aligned}$$

4. Evaluate $\int_5^6 \int_3^4 \int_1^2 \cos(x)\cos(y)\cos(z) dzdydx$.

$$\begin{aligned} \int_5^6 \int_3^4 \int_1^2 \cos x \cos y \cos z dzdydx &= \int_5^6 \int_3^4 \cos x \cos y \sin z \Big|_1^2 dydx \\ &= \int_5^6 \int_3^4 \cos x \cos y (\sin 2 - \sin 1) dydx = \int_5^6 \cos x \sin y (\sin 2 - \sin 1) \Big|_3^4 dx \\ &= \int_5^6 \cos x (\sin 4 - \sin 3) (\sin 2 - \sin 1) dx = \sin x (\sin 4 - \sin 3) (\sin 2 - \sin 1) \Big|_5^6 \\ &= (\sin 6 - \sin 5) (\sin 4 - \sin 3) (\sin 2 - \sin 1) \end{aligned}$$

5. Use a triple integral to find the volume of the great pyramid of Cheops given that it is 482 feet tall and has a square base that is 754 feet on each side. (HINT: Take advantage of symmetry.)

$$\begin{aligned} \text{volume} &= 8 \int_0^{377} \int_0^x \int_0^{-\frac{482}{377}x+482} dzdydx = 8 \int_0^{377} \int_0^x \left(-\frac{482}{377}x + 482 \right) dydx \\ &= 8 \int_0^{377} \left(-\frac{482}{377}x + 482 \right) y \Big|_0^x dx = 8 \int_0^{377} \left(-\frac{482}{377}x^2 + 482x \right) dx \\ &= 8 \left(-\frac{482}{377} \frac{x^3}{3} + 241x^2 \right) \Big|_0^{377} = 8 \left(-\frac{482 \cdot 377^2}{3} + 241 \cdot 377^2 \right) \\ &= 91,341,570.67 \text{ ft}^3 \end{aligned}$$