SURFACE INTEGRALS - ANSWERS

In each problem below you are given a surface *S*, defined by $z = f(x,y)$, over a region *R*, defined by the given limits on *x* and *y*. You are also given a function $g = g(x,y,z)$. Use the relation $dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA = \sqrt{z_x^{\,2} + z_y^{\,2} + 1}\, dA$ to find the value of the surface integral of *g* on the surface *S* by evaluating

$$\text{Surface Integral } = \iint_S g(x,y,z)\,dS = \iint_R g(x,y,z)\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

$$= \iint_R g(x,y,z)\sqrt{z_x^{\,2} + z_y^{\,2} + 1}\, dA.$$

1. $S: z = y$
   $R: 0 \le x \le 1,\ 0 \le y \le 2$
   $g(x,y,z) = x + y + z$

$$g(x,y,z) = x + y + z = x + 2y$$

$$dS = \sqrt{z_x^{\,2} + z_y^{\,2} + 1}\, dA = \sqrt{2}\, dA$$

$$\iint_S g\, dS = \iint_R (x+2y)\cdot\sqrt{2}\, dA = \sqrt{2}\int_0^1\int_0^2 (x+2y)\, dy dx$$

$$= \sqrt{2}\int_0^1 xy + y^2 \Big|_0^2 dx = \sqrt{2}\int_0^1 (2x+4)\, dx = \sqrt{2}\cdot(x^2+4x)\Big|_0^1 = 5\sqrt{2}$$

2. $S: z = x^2 + y^2 + 1$

$R: 0 \le x \le 1,\ 0 \le y \le 1$

$g(x,y,z) = \dfrac{z}{\sqrt{4x^2 + 4y^2 + 1}}$

$$g(x,y,z) = \frac{z}{\sqrt{4x^2 + 4y^2 + 1}} = \frac{x^2 + y^2 + 1}{\sqrt{4x^2 + 4y^2 + 1}}$$

$$dS = \sqrt{z_x{}^2 + z_y{}^2 + 1}\, dA = \sqrt{4x^2 + 4y^2 + 1}\, dA$$

$$\iint_S g\, dS = \iint_R \frac{x^2 + y^2 + 1}{\sqrt{4x^2 + 4y^2 + 1}} \cdot \sqrt{4x^2 + 4y^2 + 1}\, dA = \int_0^1 \int_0^1 (x^2 + y^2 + 1)\, dy dx$$

$$= \int_0^1 x^2 y + \frac{y^3}{3} + y \Big|_0^1 dx = \int_0^1 \left(x^2 + \frac{4}{3} \right) dx = \left(\frac{x^3}{3} + \frac{4}{3}x \right) \Big|_0^1 = \frac{5}{3}$$

3. $S: z = -x - y + 2$

$R: 0 \le x \le \dfrac{\pi}{2},\ 0 \le y \le \dfrac{\pi}{2}$

$g(x,y,z) = \cos x + \sin y$

$$g(x,y,z) = \cos x + \sin y$$

$$dS = \sqrt{z_x{}^2 + z_y{}^2 + 1}\, dA = \sqrt{3}\, dA$$

$$\iint_S g\, dS = \iint_R (\cos x + \sin y) \cdot \sqrt{3}\, dA = \sqrt{3} \int_0^{\pi/2} \int_0^{\pi/2} (\cos x + \sin y)\, dy dx$$

$$= \sqrt{3} \int_0^{\pi/2} y \cos x - \cos y \Big|_0^{\pi/2} dx = \sqrt{3} \int_0^{\pi/2} \left(\frac{\pi}{2} \cos x + 1 \right) dx = \sqrt{3} \left(\frac{\pi}{2} \sin x + x \right) \Big|_0^{\pi/2}$$

$$= \sqrt{3} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = \pi \sqrt{3}$$