SPHERICAL INTEGRALS - ANSWERS

For each problem below, set up and evaluate a triple integral in spherical coordinates.

1. Let V be a sphere with center at the origin and $\text{radius} = r$. Find the volume of V.

$$\iiint_V dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{\rho^3}{3}\sin\varphi \Big|_0^r \, d\varphi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{r^3}{3}\sin\varphi \, d\varphi d\theta = \int_0^{2\pi} -\frac{r^3}{3}\cos\varphi \Big|_0^{\pi} \, d\theta = \int_0^{2\pi} -\frac{r^3}{3}(\cos\pi - \cos 0)\, d\theta$$

$$= \int_0^{2\pi} \frac{2r^3}{3} d\theta = \frac{2r^3}{3}\Big|_0^{2\pi} = \frac{4}{3}\pi r^3$$

2. Find the volume of the solid bounded by the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{1} \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{\rho^3}{3}\sin\varphi \Big|_0^1 \, d\varphi d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{1}{3}\sin\varphi \, d\varphi d\theta$$

$$= \int_0^{2\pi} -\frac{1}{3}\cos\varphi \Big|_0^{\frac{\pi}{4}} \, d\theta = \int_0^{2\pi}\left(-\frac{1}{3\sqrt{2}} + \frac{1}{3}\right) d\theta = \left(\frac{1}{3} - \frac{1}{3\sqrt{2}}\right)\theta \Big|_0^{2\pi}$$

$$= \left(\frac{1}{3} - \frac{1}{3\sqrt{2}}\right) 2\pi$$

3. Evaluate $\iiint_V \frac{1}{x^2+y^2+z^2}\,dV$ where V is the solid region between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$.

$$\iiint_V \frac{1}{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_2^{3}\frac{1}{\rho^2}\rho^2\sin\varphi\, d\rho d\varphi d\theta = \int_0^{2\pi}\int_0^{\pi}\int_2^{3}\sin\varphi\, d\rho d\varphi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\rho\sin\varphi\Big|_2^3\, d\varphi d\theta = \int_0^{2\pi}\int_0^{\pi}\sin\varphi\, d\varphi d\theta = \int_0^{2\pi}-\cos\varphi\Big|_0^{\pi}\, d\theta = \int_0^{2\pi}2d\theta$$

$$= 2\theta\Big|_0^{2\pi} = 4\pi$$

4. Find the volume of the solid bounded above by $z=1$ and below by $z=\sqrt{x^2+y^2}$.

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sec\varphi}\rho^2\sin\varphi\, d\rho d\varphi d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\frac{\rho^3}{3}\sin\varphi\Bigg|_0^{\sec\kappa} d\varphi d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\frac{\sec^3\varphi}{3}\sin\varphi d\varphi d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\frac{\sin\varphi}{3\cos^3\varphi}d\varphi d\theta = \int_0^{2\pi}\frac{1}{6\cos^2\varphi}\Bigg|_0^{\frac{\pi}{4}} d\theta = \int_0^{2\pi}\left(\frac{1}{3}-\frac{1}{6}\right)d\theta = \int_0^{2\pi}\frac{1}{6}d\theta$$

$$= \frac{\theta}{6}\Bigg|_0^{2\pi} = \frac{\pi}{3}$$

5. Find the volume of the solid region defined by $\rho=\sin\varphi$ where $0\le\theta\le 2\pi$ and $0\le\varphi\le\pi$.

$$\text{Volume } = \int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\varphi}\rho^2\sin\varphi\, d\rho d\varphi d\theta = \int_0^{2\pi}\int_0^{\pi}\frac{\rho^3}{3}\sin\varphi\Bigg|_0^{\sin\varphi} d\varphi d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}\sin^4\varphi\, d\varphi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}(\sin^2\varphi)^2\, d\varphi d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}\left(\frac{1-\cos 2\varphi}{2}\right)^2 d\varphi d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}\frac{1-2\cos 2\varphi+\cos^2 2\varphi}{4}d\varphi d\theta$$

$$= \frac{1}{12}\int_0^{2\pi}\int_0^{\pi}1-2\cos 2\varphi+\frac{1+\cos 4\varphi}{2}d\varphi d\theta = \frac{1}{12}\int_0^{2\pi}\int_0^{\pi}\left(\frac{3}{2}-2\cos 2\varphi+\frac{\cos 4\varphi}{2}\right)d\varphi d\theta$$

$$= \frac{1}{12}\int_0^{2\pi}\left(\frac{3\varphi}{2}-\sin 2\varphi+\frac{\sin 4\varphi}{8}\right)\Bigg|_0^{\pi} d\theta = \frac{\pi}{8}\int_0^{2\pi}d\theta = \frac{\pi\theta}{8}\Bigg|_0^{2\pi} = \frac{\pi^2}{4}$$