SECOND PARTIALS TEST - ANSWERS

For each of the functions below, find the critical points and the determinant of the second partials matrix, and classify each critical point as resulting in a local maximum, local minimum, saddle point, or inconclusive. Furthermore, for each critical point $(a,b)$, specify the coordinates $\left(a,b,f(a,b)\right)$.

1. $z = f\left(x,y\right) = x^2 + y^2$

$$z_x = 2x$$
$$z_y = 2y$$

$$\begin{matrix} 2x = 0 \\ 2y = 0 \end{matrix} \Rightarrow \begin{matrix} x = 0 \\ y = 0 \end{matrix} \Rightarrow \text{ critical point } = \left(0,0\right)$$

$$D = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$$

$$\left.\begin{matrix} D(0,0) = 4 > 0 \\ z_{xx}(0,0) = 2 > 0 \end{matrix}\right\} \Rightarrow \left(0,0,0\right) \text{ is a local minimum point}$$

2. $z = f\left(x,y\right) = x^2 - y^2$

$$z_x = 2x$$
$$z_y = -2y$$

$$\begin{matrix} 2x = 0 \\ -2y = 0 \end{matrix} \Rightarrow \begin{matrix} x = 0 \\ y = 0 \end{matrix} \Rightarrow \text{ critical point } = \left(0,0\right)$$

$$D = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4$$

$$D(0,0) = -4 < 0 \Rightarrow \left(0,0,0\right) \text{ is a saddle point}$$

3. $z = f(x,y) = -\left(x^2 + y^2\right)$

$z_x = -2x$
$z_y = -2y$

$$\begin{matrix} -2x = 0 \\ -2y = 0 \end{matrix} \Rightarrow \begin{matrix} x = 0 \\ y = 0 \end{matrix} \Rightarrow \text{ critical point } = (0,0)$$

$$D = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4$$

$$\left.\begin{matrix} D(0,0) = 4 > 0 \\ z_{xx}(0,0) = -2 < 0 \end{matrix}\right\} \Rightarrow (0,0,0) \text{ is a local maximum point}$$

4. $z = f(x,y) = x^3 - 6x + y^3 - 9y$

$z_x = 3x^2 - 6 = 3(x^2 - 2)$
$z_y = 3y^2 - 9 = 3(y^2 - 3)$

$$\begin{matrix} 3(x^2 - 2) = 0 \\ 3(y^2 - 3) = 0 \end{matrix} \Rightarrow \begin{matrix} x = \pm\sqrt{2} \\ y = \pm\sqrt{3} \end{matrix}$$
$$\Rightarrow \text{ critical point } = \left(-\sqrt{2},-\sqrt{3}\right),\left(\sqrt{2},\sqrt{3}\right),\left(-\sqrt{2},\sqrt{3}\right),\left(\sqrt{2},-\sqrt{3}\right)$$

$$D = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 6x & 0 \\ 0 & 6y \end{vmatrix} = 36xy$$

$$\left.\begin{matrix} D(-\sqrt{2},-\sqrt{3}) = 36\sqrt{6} > 0 \\ z_{xx}(-\sqrt{2},-\sqrt{3}) = -6\sqrt{2} < 0 \end{matrix}\right\} \Rightarrow \left(-\sqrt{2},-\sqrt{3},\approx 16.0492\right) \text{ is a local maximum point}$$

$$\left.\begin{matrix} D(\sqrt{2},\sqrt{3}) = 36\sqrt{6} > 0 \\ z_{xx}(\sqrt{2},\sqrt{3}) = 6\sqrt{2} > 0 \end{matrix}\right\} \Rightarrow \left(\sqrt{2},\sqrt{3},\approx -16.0492\right) \text{ is a local minimum point}$$

$$D(-\sqrt{2},\sqrt{3}) = -36\sqrt{6} < 0 \Rightarrow \left(-\sqrt{2},\sqrt{3},\approx -4.7355\right) \text{ is a saddle point}$$

$$D(\sqrt{2},-\sqrt{3}) = -36\sqrt{6} < 0 \Rightarrow \left(\sqrt{2},-\sqrt{3},\approx 4.7355\right) \text{ is a saddle point}$$

5. $z = f(x, y) = x^3 - 12xy - y^4$

$$z_x = 3x^2 - 12y = 3(x^2 - 4y)$$
$$z_y = -4y^3 - 12x = -4(y^3 + 3x)$$

$$\begin{matrix} 3(x^2-4y)=0 \\ -4(3x+y^3)=0 \end{matrix} \Rightarrow \begin{matrix} x^2-4y=0 \\ 3x+y^3=0 \end{matrix} \Rightarrow x = \frac{-y^3}{3} \Rightarrow x^2 = \frac{y^6}{9} \Rightarrow \frac{y^6}{9} - 4y = 0$$
$$\Rightarrow y^6 - 36y = 0 \Rightarrow y(y^5 - 36) = 0 \Rightarrow y = 0 \text{ or } y = 36^{1/5}$$
$$\Rightarrow \text{ critical point } = (0,0), \left(-\frac{36^{3/5}}{3}, 36^{1/5}\right)$$

$$D = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -12 \\ -12 & -12y^2 \end{vmatrix} = -72xy^2 - 144$$

$D(0,0) = -144 < 0 \Rightarrow (0,0,0)$ is a saddle point

$$\left.\begin{matrix} D\left(-\frac{36^{3/5}}{3}, 36^{1/5}\right) \approx 720 > 0 \\ z_{xx}\left(-\frac{36^{3/5}}{3}, 36^{1/5}\right) \approx -17.1716 < 0 \end{matrix}\right\} \Rightarrow \approx (-2.9, 2.0, 29.3) \text{ is a local maximum point}$$