The Dot Product
&
The Cross Product

Mathematicians are often a little weird.

They are frequently obsessive-compulsive.

Give them a new toy, and they immediately want to do arithmetic with it.

At one point, their new toy was “vectors.”

We've seen how to add and subtract vectors, …

But how do you multiply vectors?

In 1773, Joseph Lagrange was working on a problem involving tetrahedrons, and he came up with two ways to multiply vectors.

Today, we call these two methods the “dot product” and the “cross product.”

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Ideas are also subject to the doctrine of survival of the fittest.

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a. Extremely useful, and

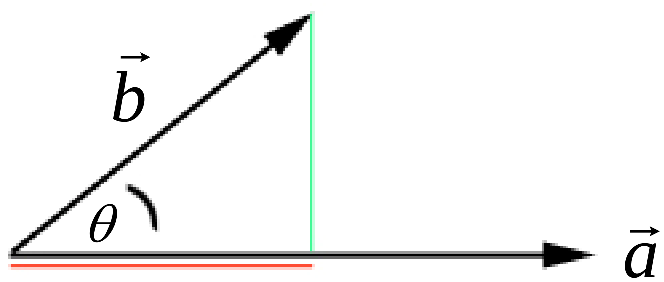
b. Rather bizarre in appearance.

The Dot Product

We'll define the dot product as follows:

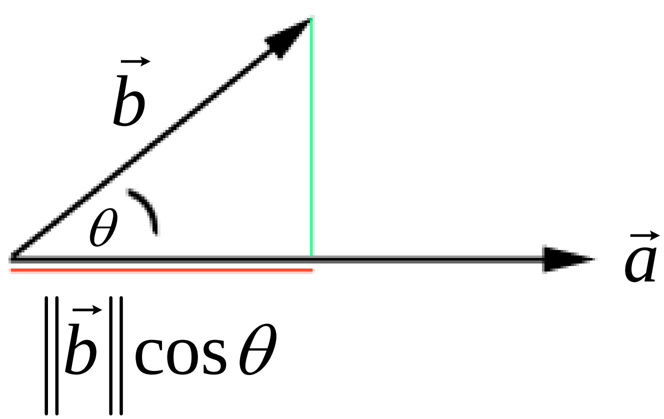
$$If\ \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\ and\ \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k},$$

$$then\ \vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta,\ where\ \theta\ is\ the\ angle\ between\ the\ vectors.$$



What does this definition mean?

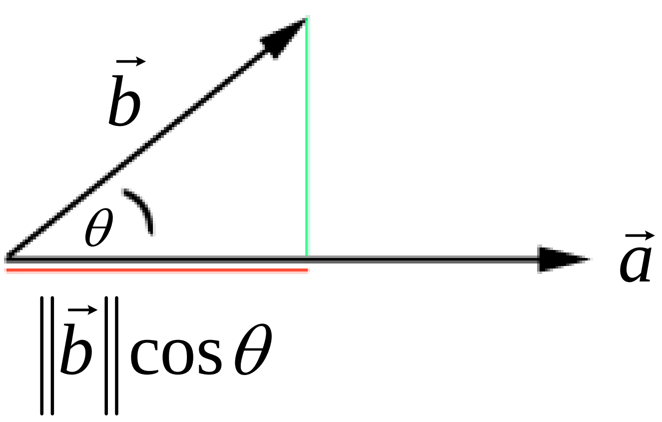
$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$



What does this definition mean?

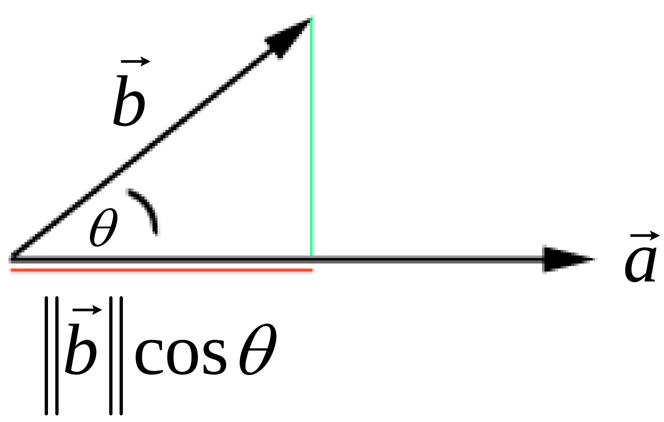
We can think of the dot product as the length of vector *a* times the length of the component of vector *b* that is parallel to vector *a.*

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$



Alternatively, we could think of it as the length of vector *b* times the length of the component of vector *a* that is parallel to vector *b.*

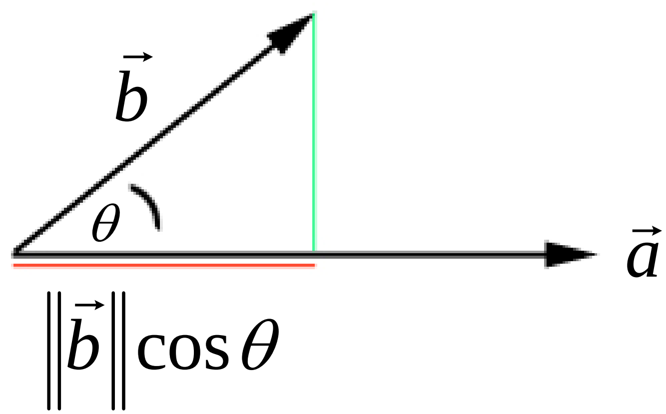
$$\vec{a} \cdot \vec{b} = \left\|\vec{a}\right\| \left\|\vec{b}\right\| \cos\theta$$



Alternatively, we could think of it as the length of vector *b* times the length of the component of vector *a* that is parallel to vector *b.*

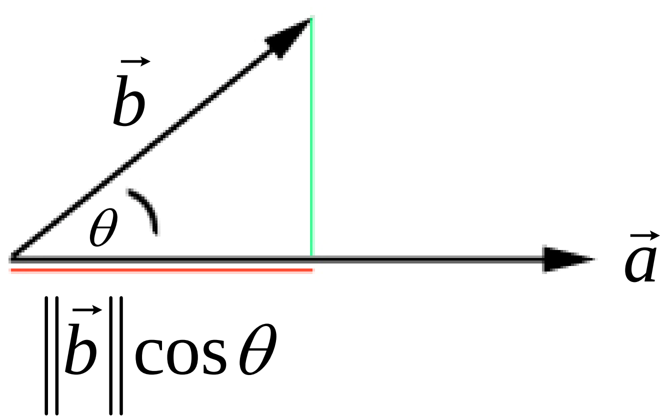
The result is the same either way.

$$\vec{a}\cdot\vec{b} = \left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta$$



The dot product gives us a way of multiplying two vectors such that the result is a number (or scalar), not another vector.

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$



The problem, though, is that we need to know the angle between the vectors in order to accomplish this.

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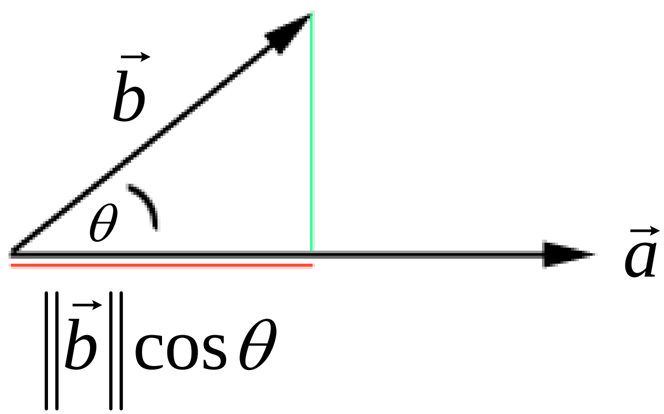
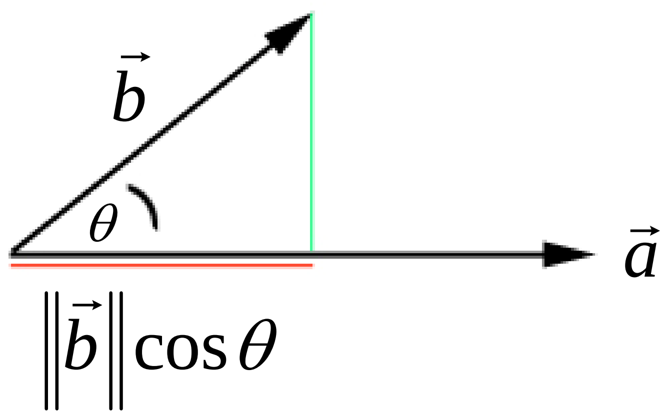


The problem, though, is that we need to know the Angle between the vectors in order to accomplish This.

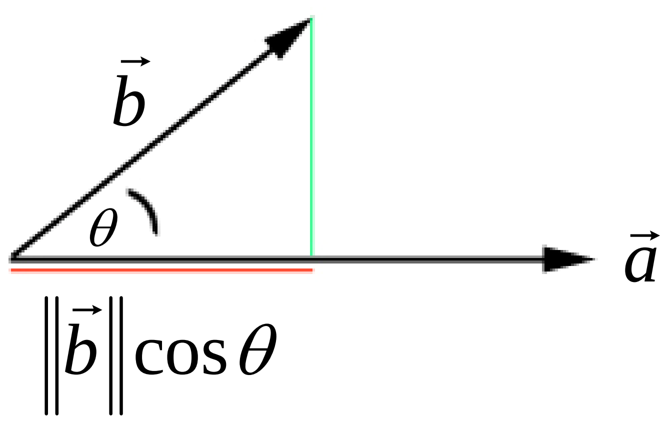
And this is something we often don't know.

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$



Fortunately, however, we have a theorem to help us.

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$



$$Theorem: \ \vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta = a_1 b_1 + a_2 b_2 + a_3 b_3 = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

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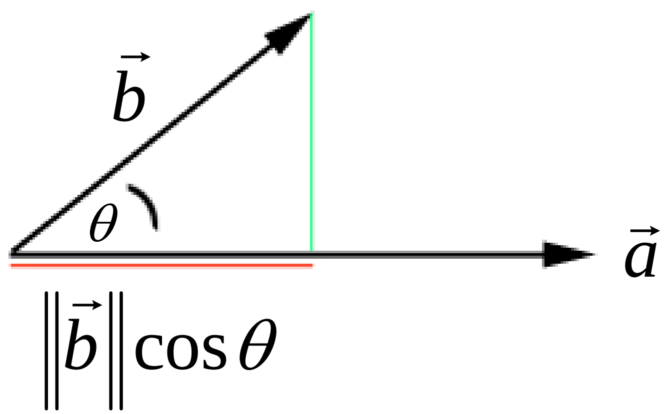
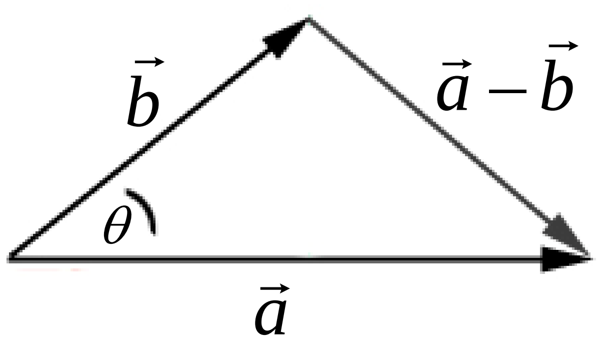


$$Theorem: \vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3 = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Proof: By the law of cosines,

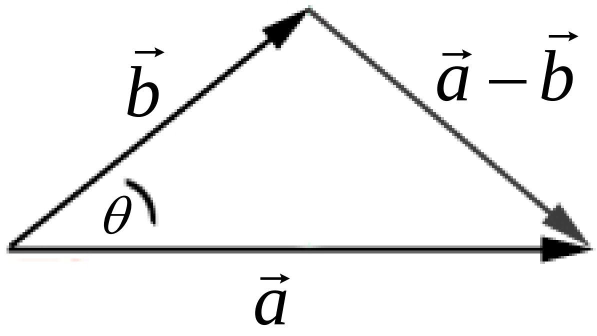
$$\|\vec{a}-\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta.$$



But this implies that,

$$(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2$$

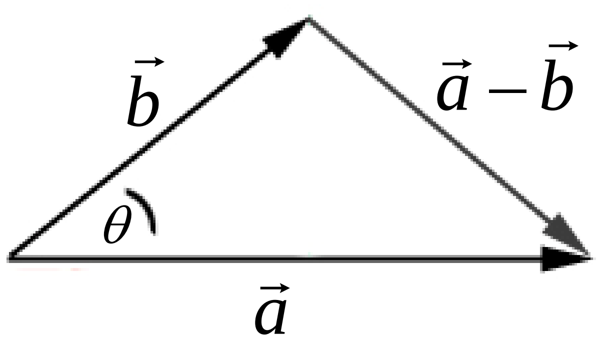
$$= a_1{}^2 + a_2{}^2 + a_3{}^2 + b_1{}^2 + b_2{}^2 + b_3{}^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta.$$



Expanding the left side of the equation yields,

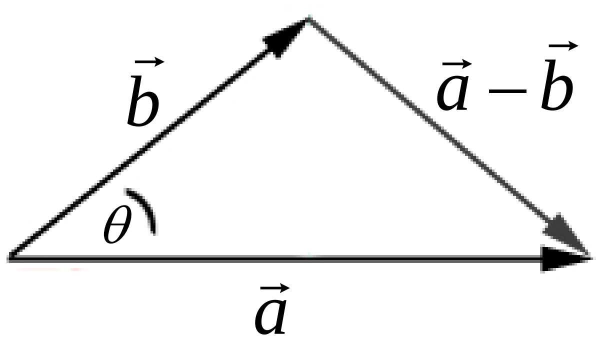
$$a_1^2 - 2a_1b_1 + b_1^2 + a_2^2 - 2a_2b_2 + b_2^2 + a_3^2 - 2a_3b_3 + b_3^2$$

$$= a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2\left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta.$$



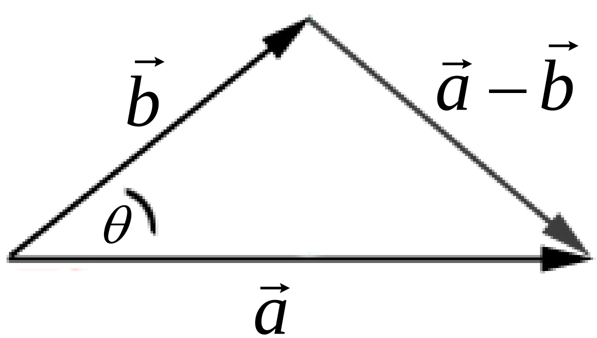
Next, subtracting like terms from each side gives,

$$-2a_1b_1 - 2a_2b_2 - 2a_3b_3 = -2\left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta.$$



And finally, we divide each side by -2.

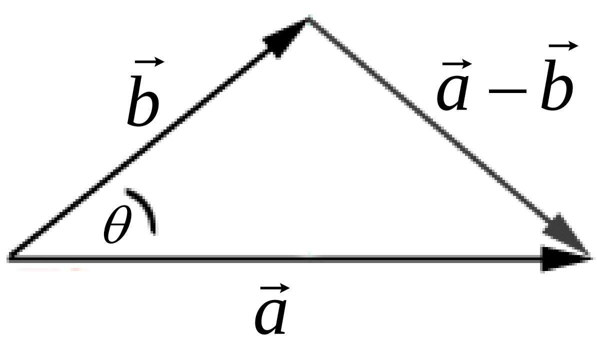
$$a_1b_1 + a_2b_2 + a_3b_3 = \|\vec{a}\|\|\vec{b}\|\cos\theta = \vec{a}\cdot\vec{b}.$$



As a corollary, we get the following formula for finding the angle between two vectors.

$$\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\|\vec{a}\|\|\vec{b}\|} = \cos\theta.$$

$$0 \leq \theta \leq \pi$$



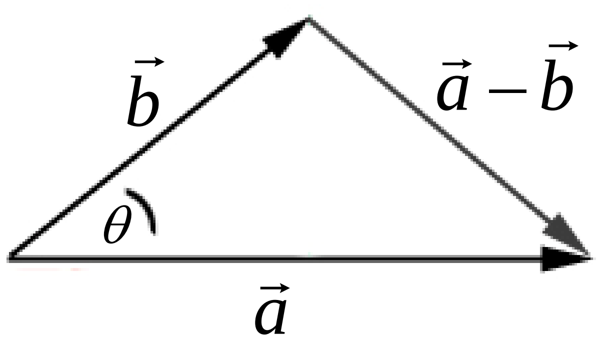
<u>EXAMPLE:</u>

$$\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\vec{b} = -\hat{i} + 2\hat{j} - 2\hat{k}$$

$$\vec{a}\cdot\vec{b} = (2)(-1) + (3)(2) + (4)(-2) = -4$$

$$\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \frac{-4}{\sqrt{4+9+16}\sqrt{1+4+4}} = \frac{-4}{3\sqrt{29}} = \cos\theta \Rightarrow \theta = \cos^{-1}\left(\frac{-4}{3\sqrt{29}}\right) \approx 104.3^\circ$$



A fairly immediate and important consequence of what we've seen is the following:

$$\vec{a}\cdot\vec{b}=\|\vec{a}\|\|\vec{b}\|\cos\theta=0 \ \textit{if and only if} \ \theta=90^\circ \ \textit{or} \ \vec{a}=\vec{0} \ \textit{or} \ \vec{b}=\vec{0}.$$

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In other words, we can determine if two vectors are perpendicular or not simply by looking to see if the dot product is zero. Furthermore, the zero vector is considered perpendicular to all vectors.

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As a final note, the dot product is also known as the "scalar product."

The Cross Product

The cross product of two vectors is defined as follows:

$$If\ \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\ and\ \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k},$$

$$then\ \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{k}$$

$$= (a_2 b_3 - a_3 . b_2)\hat{i} - (a_1 b_3 - a_3 . b_1)\hat{j} + (a_1 b_2 - a_2 . b_1)\hat{k}.$$

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$$= (a_2b_3 - a_3.b_2)\hat{i} - (a_1b_3 - a_3.b_1)\hat{j} + (a_1b_2 - a_2.b_1)\hat{k}.$$

Isn’t that cool!

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$$\vec{a}\cdot\left(\vec{a}\times\vec{b}\right) = a_1(a_2b_3 - a_3.b_2) + a_2\left[-(a_1b_3 - a_3.b_1)\right] + a_3(a_1b_2 - a_2.b_1)$$

$$= a_1a_2b_3 - a_1a_3b_2 - a_1a_2b_3 + a_2a_3b_1 + a_1a_3b_2 - a_2a_3b_1 = 0$$

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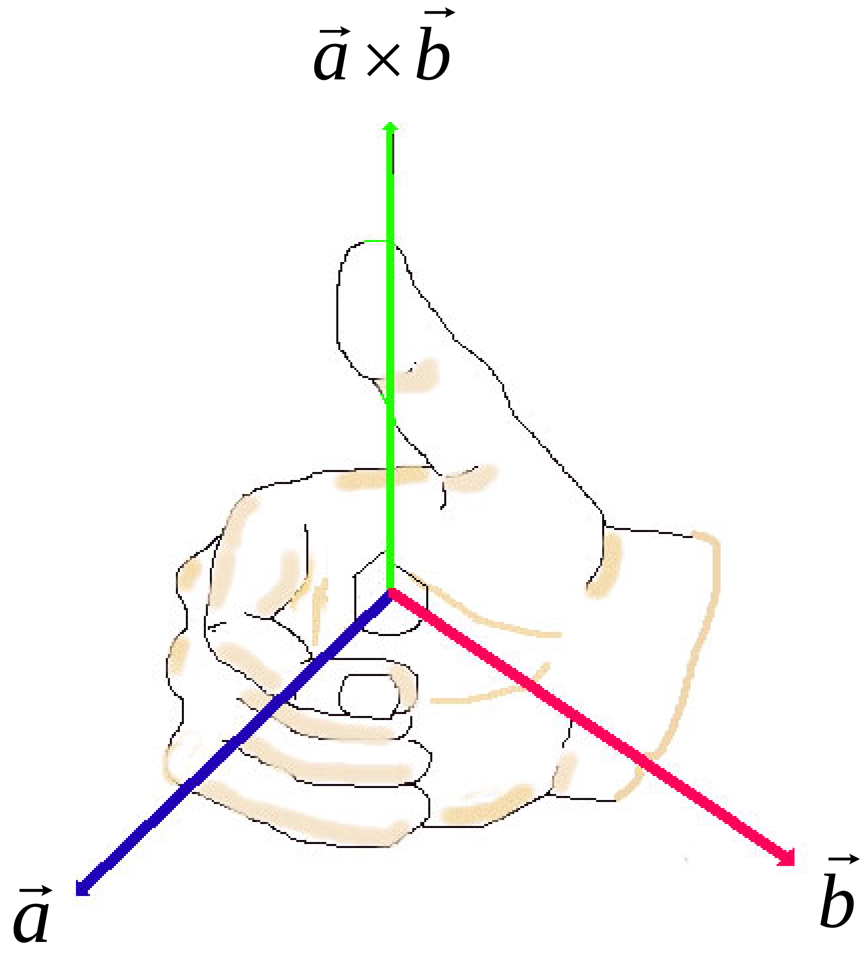
The proof that the cross product is perpendicular to vector *b* is similar.

The cross product, $\vec{a} \times \vec{b}$, obeys a right-hand rule.



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One of the more important ones is below:

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The proof is amazingly simple!

Proof:

$$\left\|\vec{a}\times\vec{b}\right\|^2 = (a_2b_3 - a_3.b_2)^2 + (a_1b_3 - a_3.b_1)^2 + (a_1b_2 - a_2.b_1)^2$$

$$= a_2{}^2b_3{}^2 - 2a_2a_3b_2b_3 + a_3{}^2b_2{}^2 + a_1{}^2b_3{}^2 - 2a_1a_3b_1b_3 + a_3{}^2b_1{}^2 + a_1{}^2b_2{}^2 - 2a_1a_2b_1b_2 + a_2{}^2b_1{}^2$$

$$= a_1{}^2b_2{}^2 + a_1{}^2b_3{}^2 + a_2{}^2b_1{}^2 + a_2{}^2b_3{}^2 + a_3{}^2b_1{}^2 + a_3{}^2b_2{}^2 - 2a_1a_2b_1b_2 - 2a_1a_3b_1b_3 - 2a_2a_3b_2b_3$$

$$= a_1{}^2b_1{}^2 + a_2{}^2b_2{}^2 + a_3{}^2b_3{}^2 + a_1{}^2b_2{}^2 + a_1{}^2b_3{}^2 + a_2{}^2b_1{}^2 + a_2{}^2b_3{}^2 + a_3{}^2b_1{}^2 + a_3{}^2b_2{}^2 - 2a_1a_2b_1b_2 - 2a_1a_3b_1b_3 - 2a_2a_3b_2b_3 - a_1{}^2b_1{}^2 - a_2{}^2b_2{}^2 - a_3{}^2b_3{}^2$$

$$= (a_1{}^2 + a_2{}^2 + a_3{}^2)(b_1{}^2 + b_2{}^2 + b_3{}^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

$$= \left\|\vec{a}\right\|^2 \left\|\vec{b}\right\|^2 - \left(\vec{a}\cdot\vec{b}\right)^2 = \left\|\vec{a}\right\|^2 \left\|\vec{b}\right\|^2 - \left(\left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta\right)^2$$

$$= \left\|\vec{a}\right\|^2 \left\|\vec{b}\right\|^2 - \left\|\vec{a}\right\|^2 \left\|\vec{b}\right\|^2 \cos^2\theta = \left\|\vec{a}\right\|^2 \left\|\vec{b}\right\|^2 \left(1 - \cos^2\theta\right) = \left\|\vec{a}\right\|^2 \left\|\vec{b}\right\|^2 \sin^2\theta.$$

Thus,

$$\left\|\vec{a}\times\vec{b}\right\|^2 = \left\|\vec{a}\right\|^2 \left\|\vec{b}\right\|^2 \sin^2\theta.$$

And since,

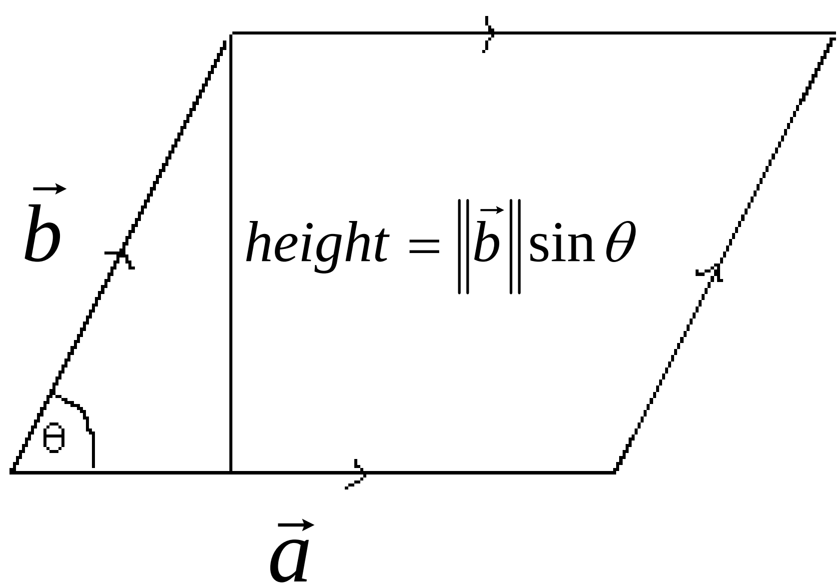
$$\sin\theta \geq 0 \;\; for\; 0 \leq \theta \leq \pi,$$

We have that,

$$\left\|\vec{a}\times\vec{b}\right\| = \left\|\vec{a}\right\|\left\|\vec{b}\right\|\sin\theta.$$

One application of this result is a formula for the area of a parallelogram.

$$Area = \|\vec{a}\|\|\vec{b}\|\sin\theta = \|\vec{a} \times \vec{b}\|$$



EXAMPLE (Cross Product):

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$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ -1 & 2 & -2 \end{vmatrix} = -14\hat{i} + 0\hat{j} + 7\hat{k} = -14\hat{i} + 7\hat{k}$$

EXAMPLE (Area of aParallelogram):

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$$Area = \left\| \vec{a} \times \vec{b} \right\| = \left\| -14\hat{i} + 7\hat{k} \right\| = \sqrt{245} = 7\sqrt{5} \approx 15.65$$