

# VECTOR FIELDS

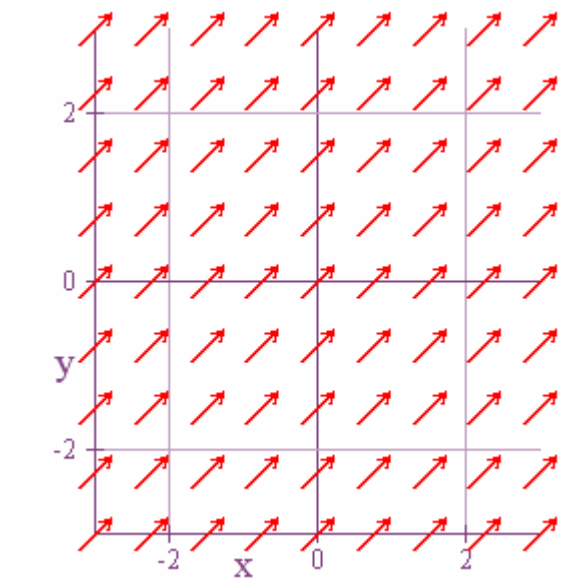


A **vector field** is a function that assigns a vector to a point in  $n$ -dimensional space. We'll generally restrict the number of dimensions to 2.

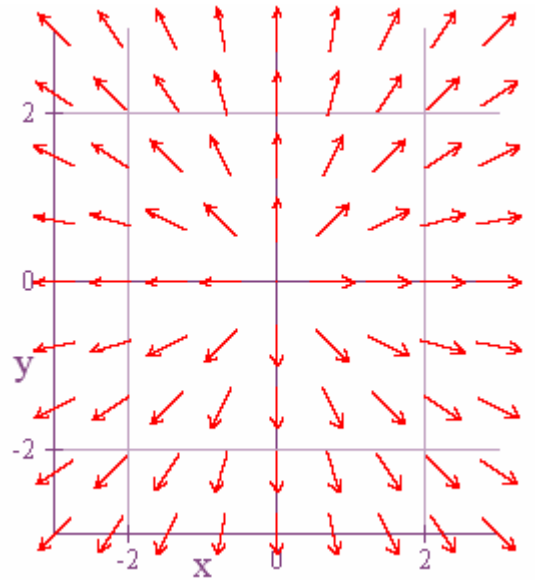
An easy way to get a vector field from a function of several variables is by finding its gradient. The resulting vector field is called a **gradient field**, and the multivariable function that gives rise to it is called a **potential** or **potential function**.

Vector fields are ideal for modeling situations where forces are present at different points in space.

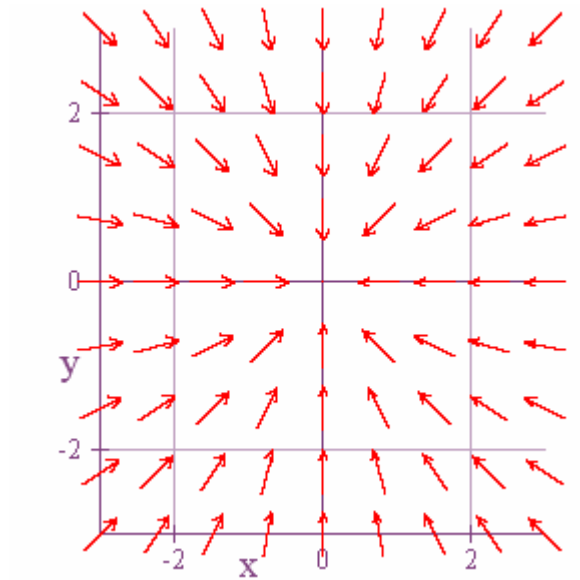
EXAMPLE:  $\vec{F}(x, y) = \hat{i} + \hat{j}$



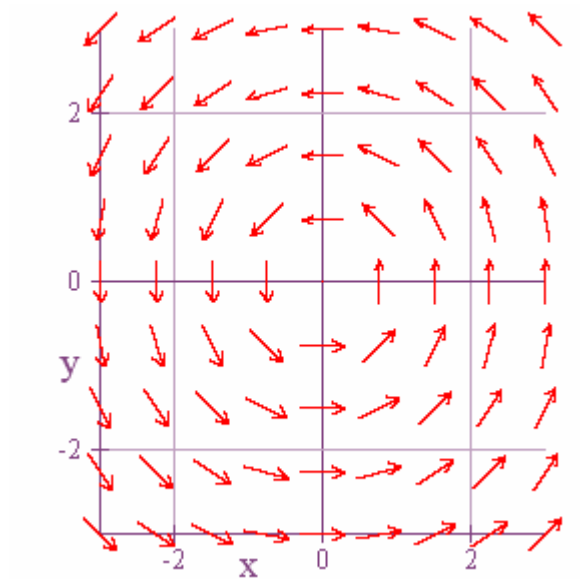
EXAMPLE:  $\vec{F}(x, y) = x\hat{i} + y\hat{j}$



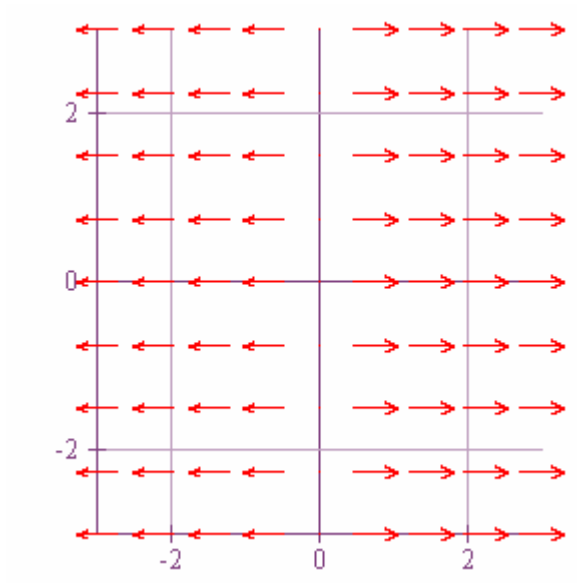
EXAMPLE:  $\vec{F}(x, y) = -x\hat{i} - y\hat{j}$



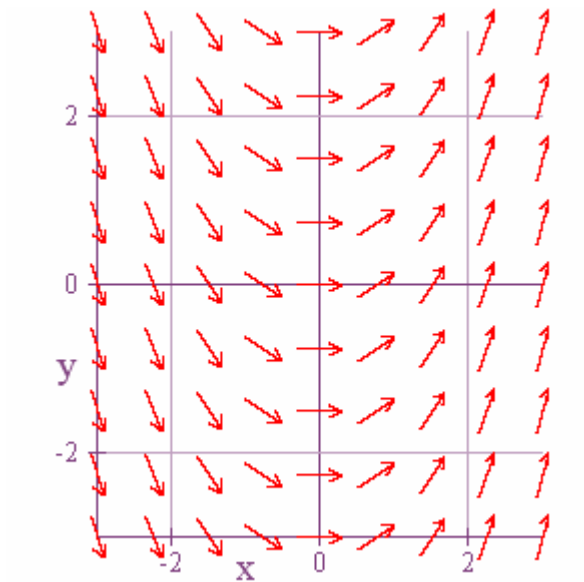
EXAMPLE:  $\vec{F}(x, y) = -y\hat{i} + x\hat{j}$



EXAMPLE:  $\vec{F}(x, y) = x\hat{i} + 0\hat{j} = x\hat{i}$



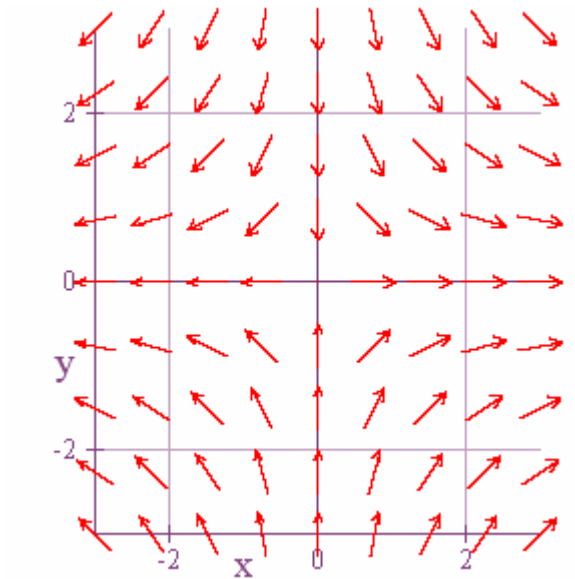
EXAMPLE:  $\vec{F}(x, y) = \hat{i} + x\hat{j}$



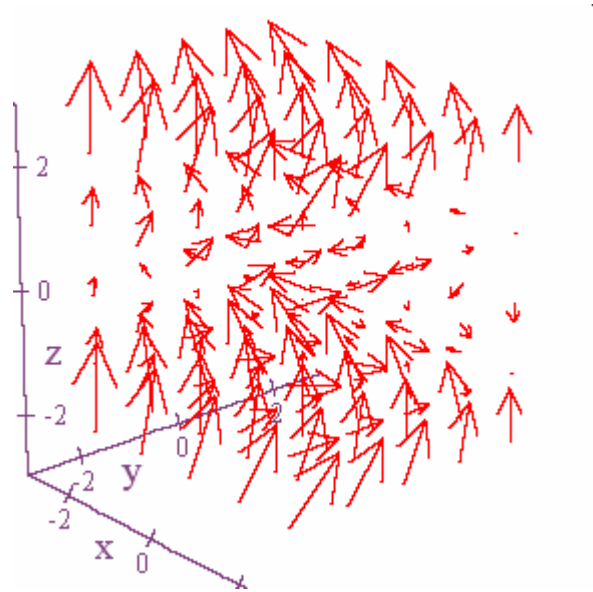


EXAMPLE:  $z = f(x, y) = x^2 - y^2$

$$\nabla f(x, y) = \vec{F}(x, y) = 2x\hat{i} - 2y\hat{j}$$



EXAMPLE:  $\vec{F}(x, y, z) = x\hat{i} - y\hat{j} + z^2\hat{k}$



Some vector fields have a tendency to cause **circulation** about a point.

We can measure this through something we call the **curl of the vector field**.

We'll find out later why this is a meaningful way to do it.

If  $\vec{F} = P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}$ , then

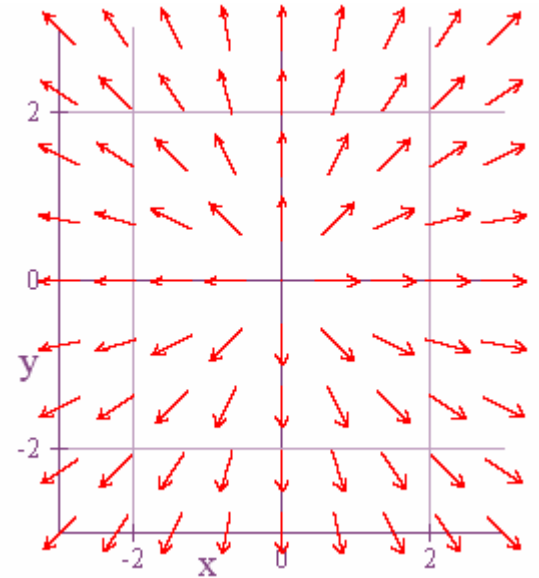
$$\begin{aligned} \text{curl of } \vec{F} = \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix} \hat{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix} \hat{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \hat{k} \\ &= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} - \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \hat{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} \end{aligned}$$

If  $\vec{F}$  is a 2-dimensional vector field,  $\vec{F} = P\hat{i} + Q\hat{j}$ , then

$$\text{curl of } \vec{F} = \nabla \times \vec{F} = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}$$

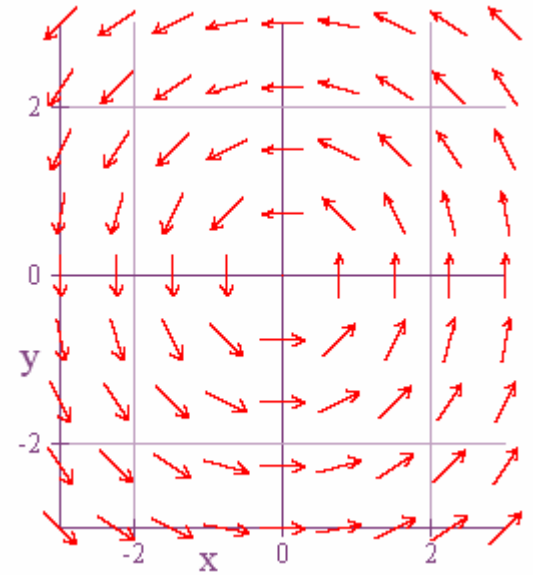
EXAMPLE: Consider  $\vec{F}(x, y) = x\hat{i} + y\hat{j}$ . There is no tendency for the field to cause circulation around any point. Thus, the curl is equal to  $\vec{0}$ .

$$\text{curl} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = \left( \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) \hat{k} = 0\hat{k} = \vec{0}$$



EXAMPLE: Consider  $\vec{F}(x, y) = -y\hat{i} + x\hat{j}$ . This vector field does create circulation about points.

$$\text{curl} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \left( \frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right) \hat{k} = 2\hat{k}$$



Other vector fields have a tendency to cause a flow or **flux** across a circular boundary. We can measure this through something we call the **divergence of the vector field**. Again, we'll find out later why our definition is meaningful.

If  $\vec{F} = P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}$ , then

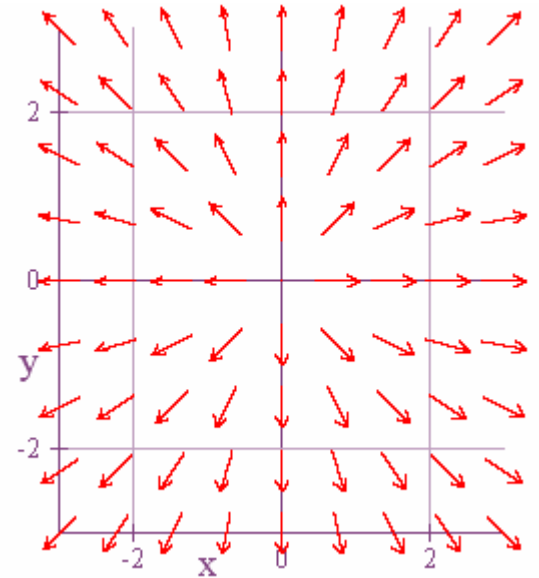
$$\begin{aligned} \text{divergence of } \vec{F} &= \nabla \cdot \vec{F} \\ &= \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \cdot (P\hat{i} + Q\hat{j} + R\hat{k}) \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \end{aligned}$$

If  $\vec{F}$  is a 2-dimensional vector field,  $\vec{F} = P\hat{i} + Q\hat{j}$ , then

$$\text{divergence of } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

EXAMPLE: Consider  $\vec{F}(x, y) = x\hat{i} + y\hat{j}$ . There is a tendency for the field to cause a flux across a circular boundary.

$$\text{divergence} = \nabla \cdot \vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 1 + 1 = 2$$



EXAMPLE: Consider  $\vec{F}(x, y) = -y\hat{i} + x\hat{j}$ . There is no tendency for the field to cause a flux across a circular boundary.

Thus, the divergence should be zero.

$$\text{divergence} = \nabla \cdot \vec{F} = \frac{\partial(-y)}{\partial x} + \frac{\partial x}{\partial y} = 0 + 0 = 0$$

