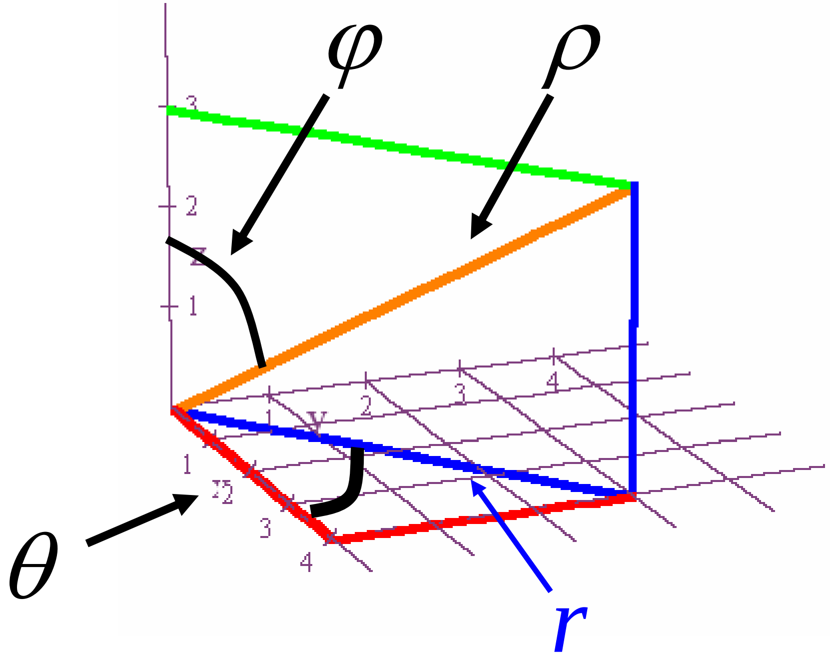
INTEGRALS IN SPHERICAL COORDINATES

$$(\rho, \theta, \varphi)$$



$$\rho^2 = x^2 + y^2 + z^2$$

$$\tan\theta = \frac{y}{x}$$

$$\varphi = \arccos\frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$x = r\cos(\theta) = \rho\sin(\varphi)\cos(\theta)$$

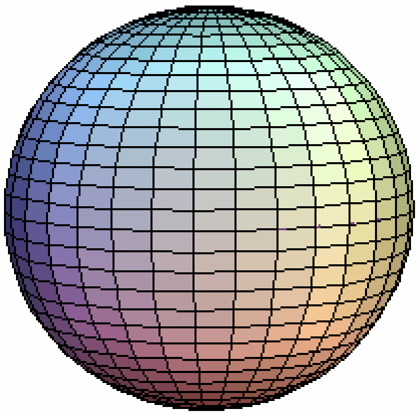
$$y = r\sin(\theta) = \rho\sin(\varphi)\sin(\theta)$$

$$z = \rho\cos(\varphi)$$

$$0 \le \rho < \infty$$

$$0 \le \varphi \le \pi$$

$$0 \le \theta < 2\pi$$

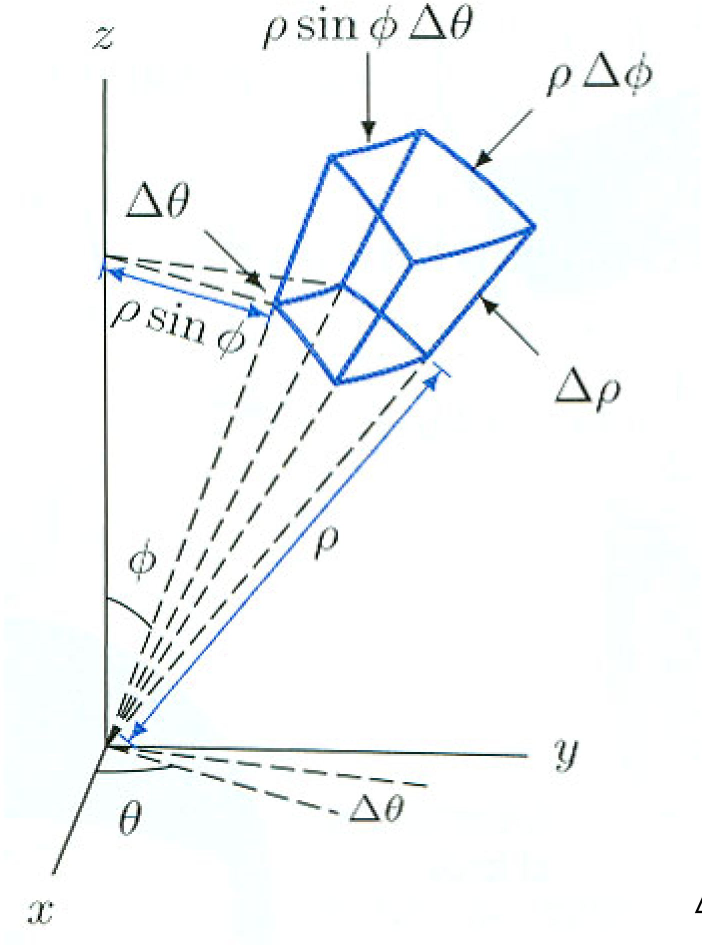


$$\rho = 5$$

$$0 \le \theta < 2\pi$$

$$0 \le \varphi \le \pi$$

TRIPLE INTEGRALS IN SPHERICAL COORDINATES



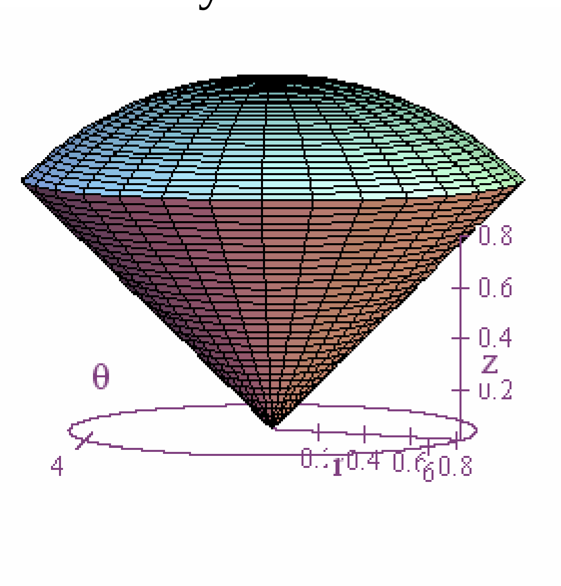
$$\Delta V \approx \Delta\rho \cdot \rho\Delta\varphi \cdot r\Delta\theta = \Delta\rho \cdot \rho\Delta\varphi \cdot \rho\sin\varphi\Delta\theta$$

$$= \rho^2 \sin\varphi\, \Delta\rho\Delta\varphi\Delta\theta$$

$$dV = \rho^2 \sin\varphi \;\, d\rho\; d\varphi\; d\theta$$

EXAMPLE: Find the volume of the ice cream cone defined by

$$0 \le \theta \le 2\pi, 0 \le \varphi \le \frac{\pi}{4}, 0 \le \rho \le \sqrt{2}.$$



$$\text{Volume } = \iiint_R dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^3}{3}\sin\varphi \Big|_0^{\sqrt{2}} d\varphi d\theta = \int_0^{2\pi}\int_0^{\pi/4} \frac{2^{3/2}}{3}\sin\varphi \, d\varphi d\theta$$

$$= \int_0^{2\pi} \frac{2^{3/2}}{3}(-\cos\varphi)\Big|_0^{\pi/4} d\theta = \int_0^{2\pi} \frac{2^{3/2}}{3}\left(\frac{-1}{\sqrt{2}}+1\right) d\theta$$

$$= \frac{\theta \cdot 2^{3/2}}{3}\left(\frac{-1}{\sqrt{2}}+1\right)\Big|_0^{2\pi} = \frac{2\pi}{3}\cdot 2^{3/2}\left(\frac{-1}{\sqrt{2}}+1\right) = \frac{4\pi}{3}\cdot\sqrt{2}\left(\frac{-1}{\sqrt{2}}+1\right)$$

$$= \frac{4\pi}{3}\left(\sqrt{2}-1\right)$$

EXAMPLE: Find $\iiint_R z^2\,dV$ on the region between the spheres $\rho = 1$ and $\rho = 2$.

$$0 \le \theta \le 2\pi$$
$$0 \le \varphi \le \pi$$
$$1 \le \rho \le 2$$
$$z = \rho\cos\varphi \Rightarrow z^2 = \rho^2\cos^2\varphi$$

$$\iiint_R z^2\,dV = \int_0^{2\pi}\int_0^{\pi}\int_1^{2} \rho^2\cos^2\varphi\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$\int_0^{2\pi}\int_0^{\pi}\left(\frac{\rho^5}{5}\cos^2\varphi\sin\varphi\right)\Bigg|_1^2 d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\frac{31}{5}\cos^2\varphi\sin\varphi\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\frac{31}{5}\frac{\left(-\cos^3\varphi\right)}{3}\Bigg|_0^{\pi} d\theta = \int_0^{2\pi}\frac{62}{15}\,d\theta = \frac{62\theta}{15}\Bigg|_0^{2\pi}$$

$$= \frac{62\cdot 2\pi}{15} - \frac{62\cdot 0}{15} = \frac{124\pi}{15}$$