Proof of the Second Partials Test

The Second Partials Test

Suppose (a,b) is a point such that $f_x(a,b) = 0 = f_y(a,b)$, and let

$$D = D(a,b) = f_{xx}(a,b)\,f_{yy}(a,b) - \left[f_{xy}(a,b) \right]^2 = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix}.$$

1. If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.
2. If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.
3. If $D < 0$, then (a,b,$f(a,b)$) is a saddle point.
4. If $D = 0$, then we know nothing.

PROOF:

Let $D(a,b) = f_{xx}(a,b) \cdot f_{yy}(a,b) - \left[f_{xy}(a,b) \right]^2$.

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Then $D_{\vec{u}}.f = \nabla f \bullet \vec{u} = f_x h + f_y k.$

Also, $D^2{}_{\vec{u}}\,f = D_{\vec{u}}\left(D_{\vec{u}}\,f\right) = \nabla\left(D_{\vec{u}}\,f\right)\bullet\vec{u}$

$$= \left[\left(f_{xx}h + f_{yx}k\right)\hat{i} + \left(f_{xy}h + f_{yy}k\right)\hat{j}\right]\bullet\left[h\hat{i} + k\,\hat{j}\right]$$

$$= f_{xx}h^2 + f_{yx}hk + f_{xy}hk + f_{yy}k^2$$

$$= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2.$$

We can rewrite this by completing the square.

$$D^2_{\vec{u}} f = f_{xx}h^2 + 2 f_{xy}hk + f_{yy}k^2$$

$$= f_{xx}\left(h^2 + \frac{2 f_{xy}}{f_{xx}} hk \right) + f_{yy}k^2$$

$$= f_{xx}\left(h^2 + \frac{2 f_{xy}}{f_{xx}} hk + \left[\frac{f_{xy}k}{f_{xx}} \right]^2 \right) + f_{yy}k^2 - \frac{f_{xy}{}^2 k^2}{f_{xx}}$$

$$= f_{xx}\left(h + \frac{f_{xy}k}{f_{xx}} \right)^2 + \frac{k^2}{f_{xx}}\left(f_{xx} f_{yy} - f_{xy}{}^2 \right).$$

$$D^2_{\vec{u}} f == f_{xx}\left(h + \frac{f_{xy}k}{f_{xx}}\right)^2 + \frac{k^2}{f_{xx}}\left(f_{xx}f_{yy} - f_{xy}{}^2\right).$$

If $D = f_{xx}f_{yy} - f_{xy}{}^2 > 0$ and $f_{xx} > 0$, then $D^2_{\vec{u}} f > 0$ for all unit vectors $\vec{u}$.

Consequently, any plane that passes through $z = f(x, y)$ and contains the point $(a, b, f(a, b))$ and is perpendicular to the xy – plane will result in a curve of intersection with $z = f(x, y)$ that is concave up.

Therefore, $(a, b, f(a, b))$ is a minimum point.

If $D^2_{\vec{u}} f < 0$ for all unit vectors $\vec{u}$, the argument is similar that $(a, b, f(a, b))$ is a maximum point.

Now suppose that $D = f_{xx} f_{yy} - f_{xy}{}^2 < 0,$

and recall $D^2_{\vec{u}} f = f_{xx} h^2 + 2 f_{xy} hk + f_{yy} k^2.$

Suppose $f_{xx} \neq 0,$ and note $f_{xx} D_{\vec{u}}{}^2 = f_{xx} \left(f_{xx} h^2 + 2 f_{xy} hk + f_{yy} k^2 \right)$

$$= f_{xx}{}^2 h^2 + 2 f_{xx} f_{yy} hk + f_{xx} f_{yy} k^2$$

$$= f_{xx}{}^2 h^2 + 2 f_{xx} f_{xy} hk + f_{xy}{}^2 k^2 + f_{xx} f_{yy} k^2 - f_{xy}{}^2 k^2$$

$$= \left(f_{xx} h + f_{xy} k \right)^2 + \left(f_{xx} f_{yy} - f_{xy}{}^2 \right) k^2$$

Hence, $f_{xx} D_{\vec{u}}{}^2 > 0$ when $h \neq 0$ & $k = 0,$

and $f_{xx} D_{\vec{u}}{}^2 < 0$ when $f_{xx} h + f_{xy} k = 0$ & $k \neq 0$

Therefore, $\left(a, b, f(a,b) \right)$ is a saddle point.

Also, if $f_{yy} \neq 0,$ then a similar argument may be used to arrive at the same conclusion that $\left(a, b, f(a,b) \right)$ is a saddle point.

By the way, how do we know that if $k \neq 0$, then there even exists an h such that $f_{xx}h + f_{xy}k = 0$?

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Therefore, we need a θ such that $\dfrac{-f_{xy}}{f_{xx}} = \dfrac{h}{k} = \dfrac{\cos\theta}{\sin\theta} = \cot\theta$.

Since the range of the $\cot\theta$ is all real numbers, such a θ exists.

Now suppose again that $D = f_{xx} f_{yy} - f_{xy}{}^2 < 0$,

and recall $D^2{}_{\vec{u}} f = f_{xx}h^2 + 2 f_{xy}hk + f_{yy}k^2$.

If $f_{xx} = 0 = f_{yy}$, then $D^2{}_{\vec{u}} f = f_{xx}h^2 + 2 f_{xy}hk + f_{yy}k^2$

reduces to $D^2{}_{\vec{u}} f = 2 f_{xy}hk$ and $D = -f_{xy}{}^2 < 0$.

Hence, $f_{xy} \neq 0$, and $D^2{}_{\vec{u}} f$ will have different signs for the unit vectors

$\vec{u}_1 = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$ and $\vec{u}_2 = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$.

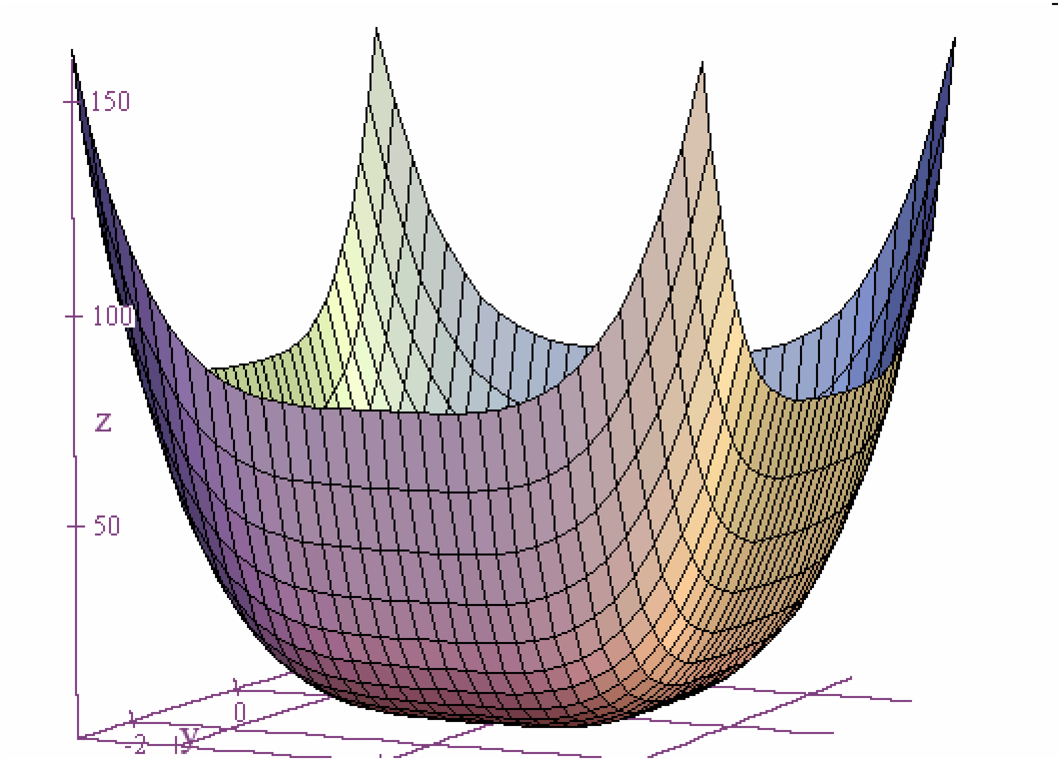
Therefore, $\left(a, b, f(a,b)\right)$ is a saddle point.

Now consider $z_1 = x^4 + y^4$ and $z_2 = x^4 - y^4$.

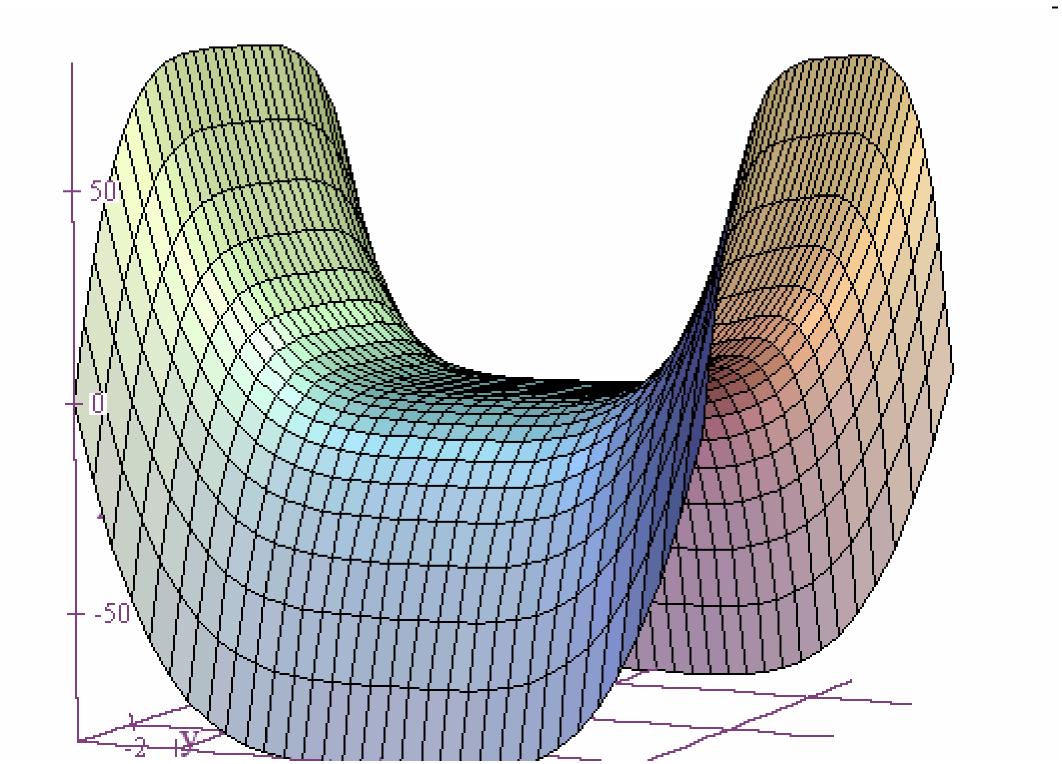
For each of these functions $(0,0)$ is a critical point.

However, it's easy to show that for each of these functions, $D(0,0) = 0$.

And from the pictures below, we see that one has a minimum point while the other has a saddle point.



$$z = x^4 + y^4$$



$$z = x^4 - y^4$$

Therefore, $D = 0$ means nothing.