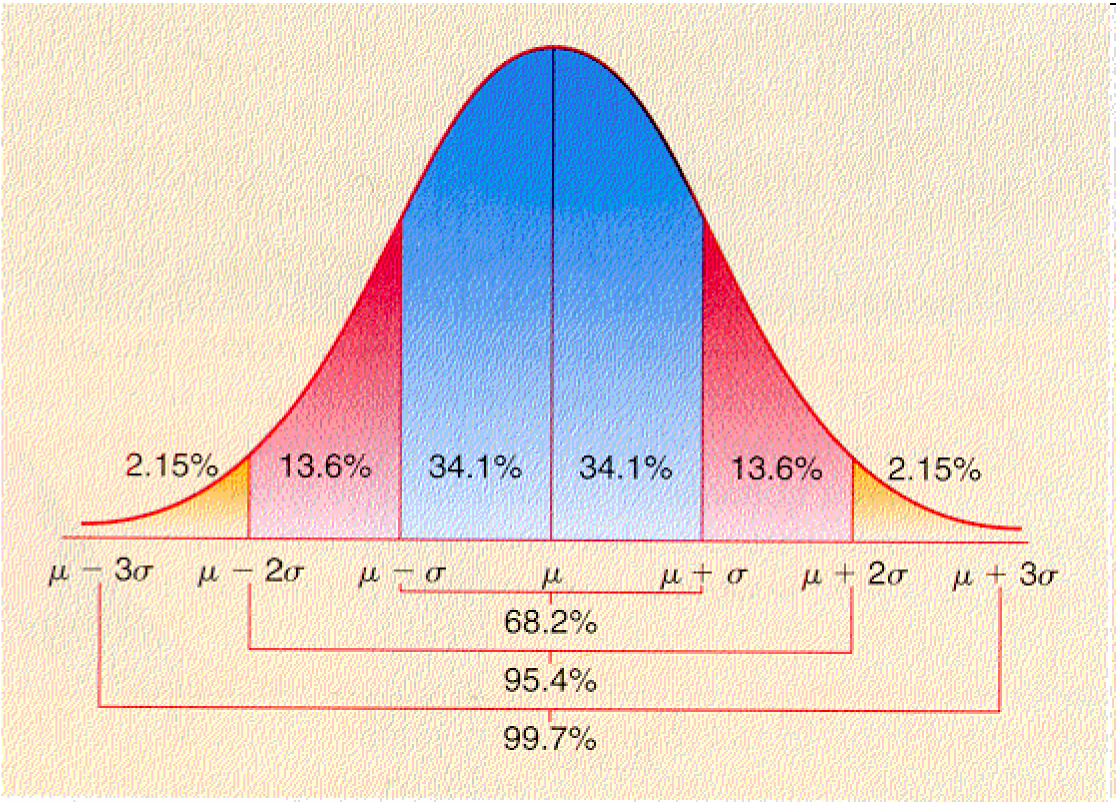
PROBABILITY DENSITY FUNCTIONS



A function $p = p(x)$ is a probability density function if:

1. $p(x) \geq 0$ for all x
2. $\int_{-\infty}^{\infty} p(x)\,dx = 1$

In this case, the probability that a value x lies between a and b is

$$P(a \leq x \leq b) = \int_a^b p(x)\,dx$$

For the normal curve, $p(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$.

The integral of this function over an interval has to be evaluated numerically.

A joint density function or probability density function in two variables is defined as a function $p(x, y)$ such that,

1. $p(x, y) \geq 0$ for all x and y
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y)\, dy dx = 1$

To find the probability that $a \leq x \leq b$ and $c \leq y \leq d$, we evaluate

$$P(a \leq x \leq b, c \leq y \leq d) = \int_a^b \int_c^d p(x, y)\, dy dx.$$

EXAMPLE: Verify that $p(x,y)=\begin{cases} x+y & \text{for } 0\le x\le 1 \text{ and } 0\le y\le 1 \\ 0 & \text{otherwise} \end{cases}$

is a joint density function.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dy\,dx=\int_0^1\int_0^1 (x+y)\,dy\,dx=\int_0^1\left(xy+\frac{y^2}{2}\right)\Bigg|_0^1 dx$$

$$=\int_0^1\left(x+\frac{1}{2}\right)dx=\left(\frac{x^2}{2}+\frac{x}{2}\right)\Bigg|_0^1=\frac{1}{2}+\frac{1}{2}=1$$

Additionally, $p(x,y)\ge 0$ for all x and y. Therefore, $p(x,y)$ is a joint density function (probability density function in two variables).

EXAMPLE: For $p(x,y)=\begin{cases} x+y & \text{for } 0\le x\le 1 \text{ and } 0\le y\le 1 \\ 0 & \text{otherwise} \end{cases}$,

find the probability that $0\le x\le 1/2$ and $0\le y\le 1/2$.

$$P(0\le x\le 1/2, 0\le y\le 1/2)=\int_0^{1/2}\int_0^{1/2}(x+y)\,dydx=\int_0^{1/2}\left(xy+\frac{y^2}{2}\right)\Bigg|_0^{1/2}dx$$

$$=\int_0^{1/2}\left(\frac{x}{2}+\frac{1}{8}\right)dx=\left(\frac{x^2}{4}+\frac{x}{8}\right)\Bigg|_0^{1/2}=\frac{1}{16}+\frac{1}{16}=\frac{1}{8}$$

THEOREM: If $p(x)$ and $q(y)$ are both probability density functions of one variable, then $f(x,y) = p(x)q(y)$ is a joint density function.

PROOF: It suffices to show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x)q(y)\,dydx = 1.$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x)q(y)\,dydx = \int_{-\infty}^{\infty} p(x)\left(\int_{-\infty}^{\infty} q(y)\,dy\right)dx = \int_{-\infty}^{\infty} p(x)\cdot 1dx$$

$$= \int_{-\infty}^{\infty} p(x)\,dx = 1.$$