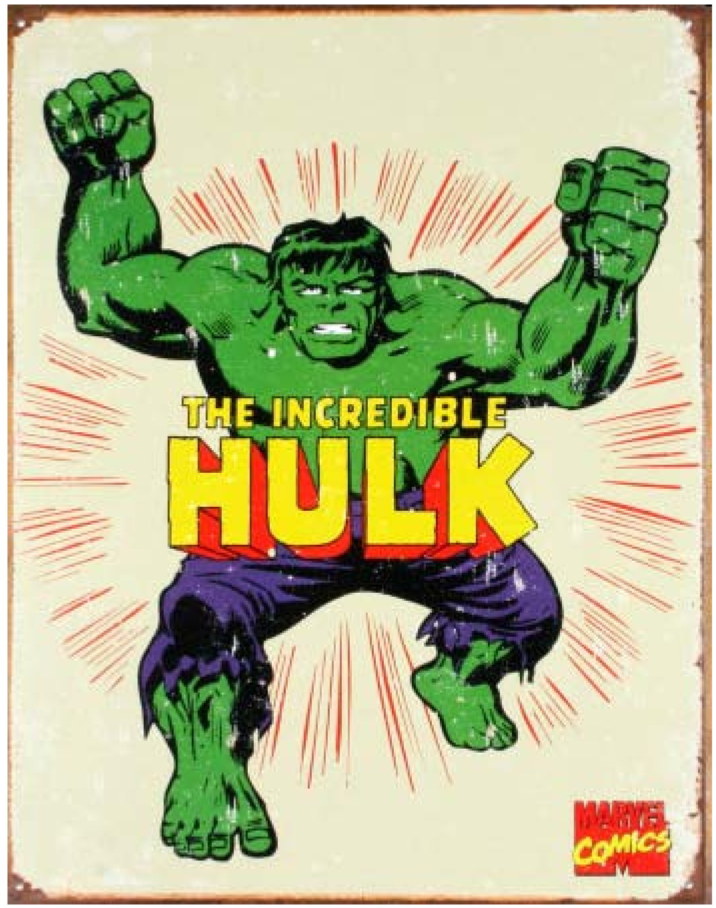
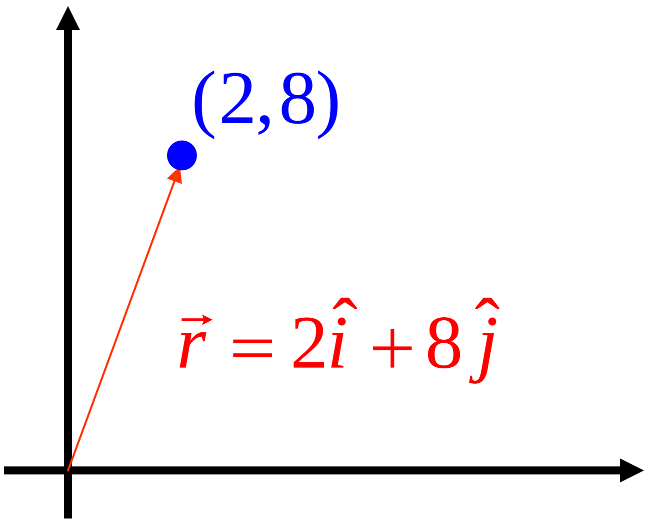
PARAMETRIC SURFACES AND TRANSFORMATIONS



A point in space can be associated with the position vector that terminates at that point.

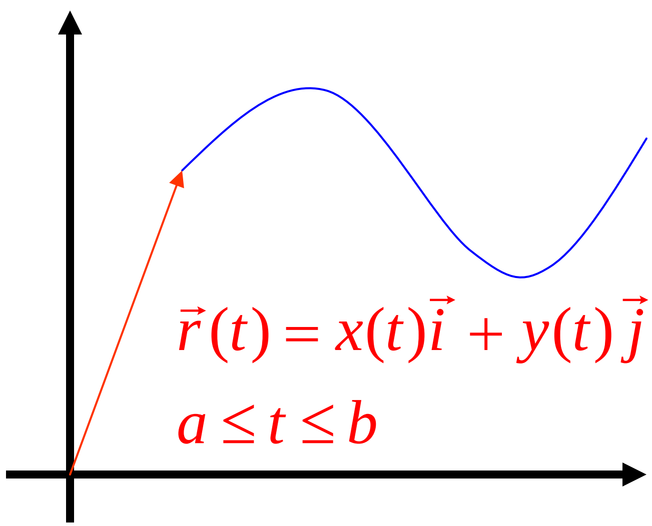


Similarly, a parametrized curved can be associated with a corresponding vector-valued function.

$$x = x(t)$$

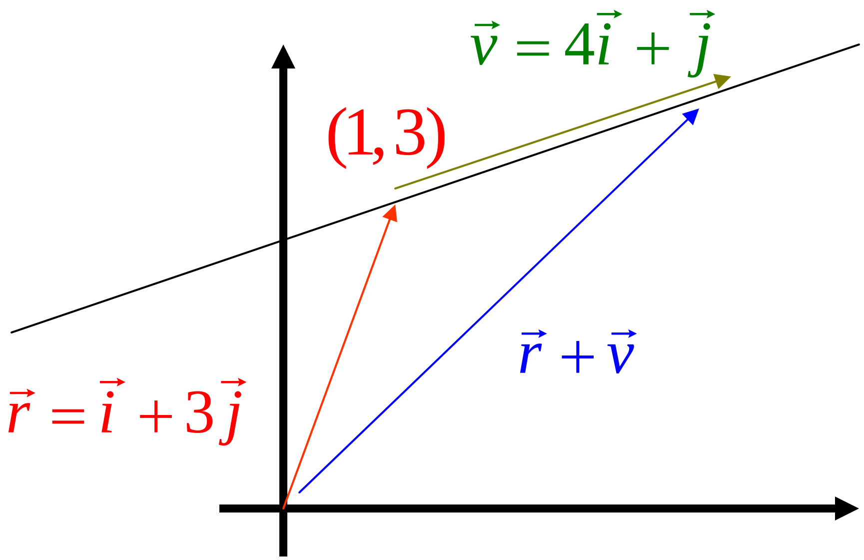
$$y = y(t)$$

$$a \le t \le b$$



Here’s how we can express a line with vectors.

$$L = \vec{r} + t\vec{v} = (1 + 4t)\vec{i} + (3 + t)\vec{j}$$



$$x = 1 + 4t$$

$$y = 3 + t$$

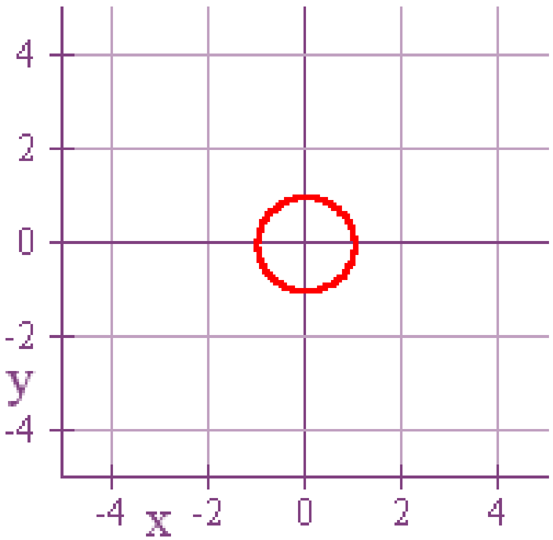
$$-\infty < t < \infty$$

Below are two ways we can describe the unit circle.

$$x = \cos t$$

$$y = \sin t$$

$$0 \le t \le 2\pi$$



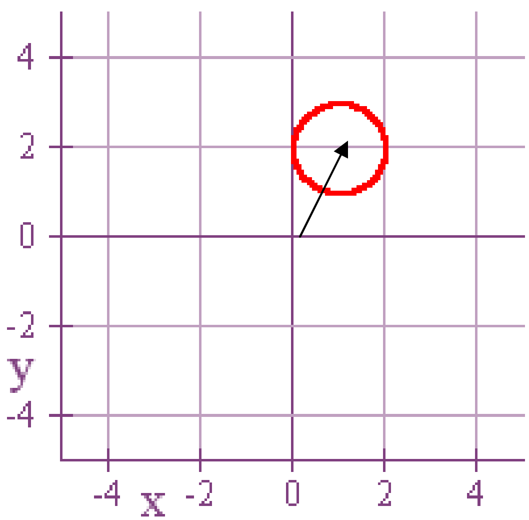
$$\vec{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j}$$

$$0 \le t \le 2\pi$$

To shift the center of this circle to another location, think in terms of adding a fixed vector to the one that describes the circle.

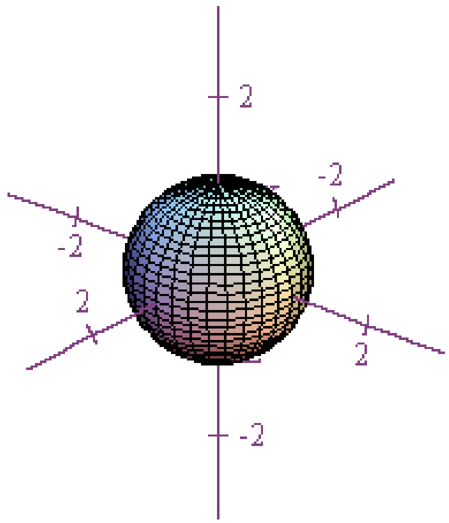
$$x = 1 + \cos t$$
$$y = 2 + \sin t$$
$$0 \le t \le 2\pi$$

$$\vec{v} = \hat{i} + 2\hat{j}$$



$$\vec{r}(t) + \vec{v} = \left(1 + \cos(t)\right)\vec{i} + \left(2 + \sin(t)\right)\vec{j}$$
$$0 \le t \le 2\pi$$

We can do this same sort of thing in three dimensions with a sphere by expressing *x*,*y*,and *z* in terms of spherical coordinates. If rho is fixed, as it is below, then our sphere is a surface described by two parameters.



$$x = \rho \sin\varphi \cos\theta$$

$$y = \rho \sin\varphi \sin\theta$$

$$z = \rho \cos\varphi$$

$$0 \le \theta \le 2\pi$$

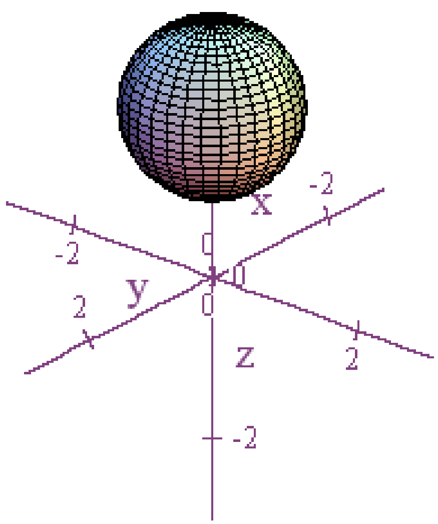
$$0 \le \varphi \le \pi$$

$$\rho = 1$$

$$\vec{r}(\theta,\varphi) = \left(\sin\varphi\cos\theta\right)\vec{i} + \left(\sin\varphi\sin\theta\right)\vec{j} + \left(\cos\varphi\right)\hat{k}$$

Again, adding a fixed vector to this will shift the center.

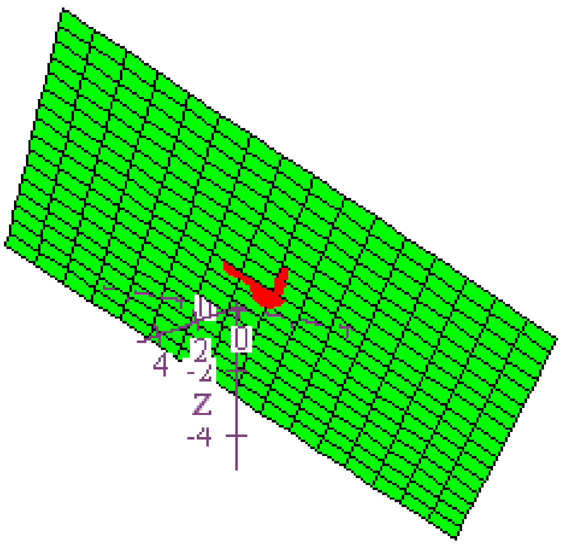
$$\vec{v} = 2\hat{k}$$



$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi + 2$$
$$0 \le \theta \le 2\pi$$
$$0 \le \varphi \le \pi$$
$$\rho = 1$$

$$\vec{r} + \vec{v} = (\sin\varphi \cos\theta)\vec{i} + (\sin\varphi \sin\theta)\vec{j} + (\cos\varphi + 2)\hat{k}$$

Planes, in general, can be described by parametric equations using a point and two non-parallel vectors that lie in the plane.



$$P = (1, 2, 1)$$

$$\vec{r} = \hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{v} = \hat{i} + \hat{j} + \hat{k}$$

$$\vec{u} = \hat{i} - \hat{j} + \hat{k}$$

$$x = 1 + s + t$$

$$y = 2 + s - t$$

$$z = 1 + s + t$$

$$\vec{r} + s\vec{v} + t\vec{u} = (1 + s + t)\hat{i} + (2 + s - t)\hat{j} + (1 + s + t)\hat{k}$$

$$-\infty < s < \infty$$

$$-\infty < t < \infty$$