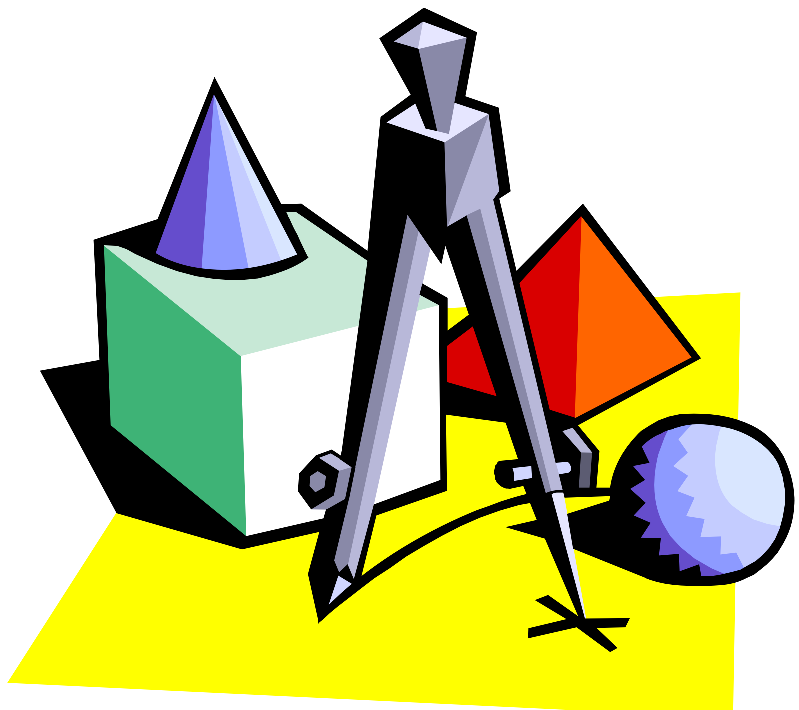
PARAMETRIC SURFACE AREA AND INTEGRALS



Instead of $z = f(x,y)$, the more natural way to describe a surface is parametrically by two parameters.

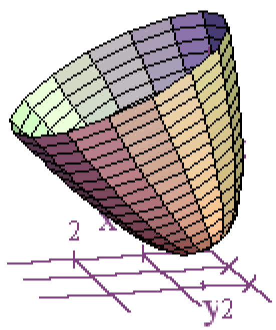
$$x = x(s,t)$$

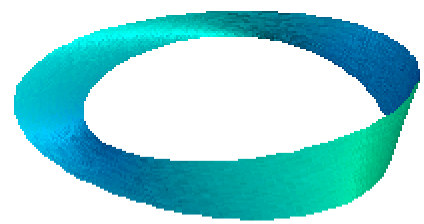
$$y = y(s,t)$$

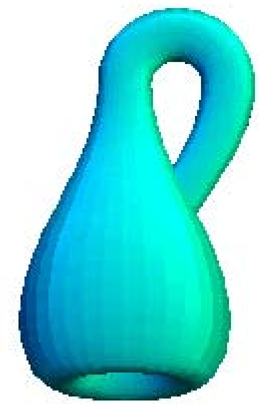
$$z = z(s,t)$$

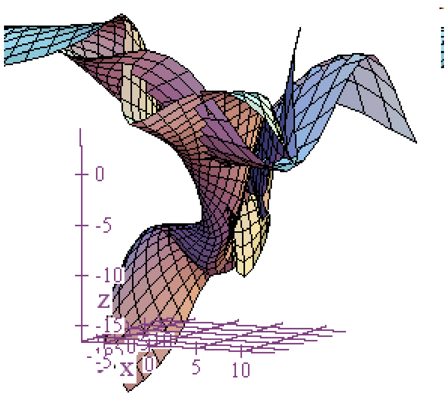
$$\vec{r}(s,t) = x(s,t)\,\hat{i} + y(s,t)\,\hat{j} + z(s,t)\,\hat{k}$$

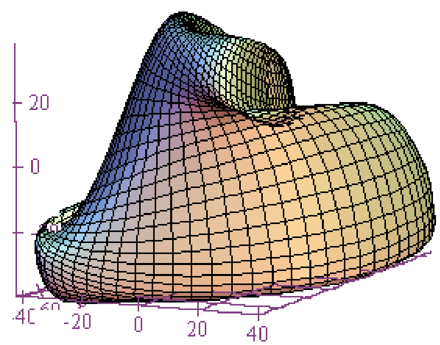
With this type of description, we can create surfaces that are quite interesting and sometimes rather bizarre.

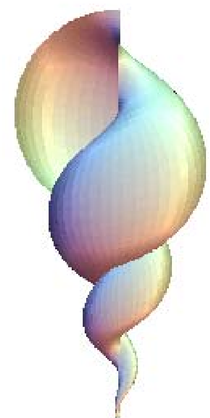




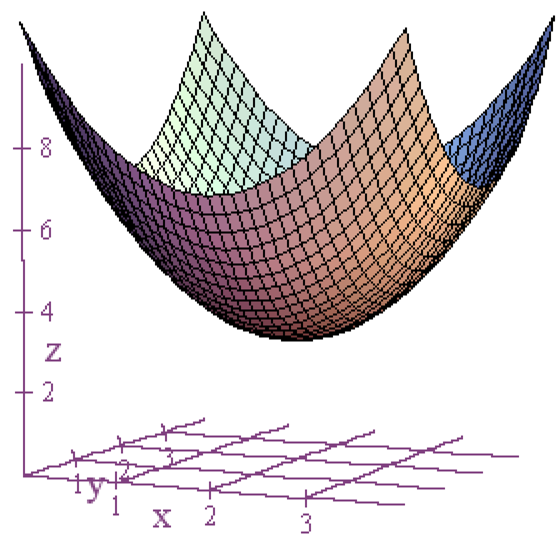






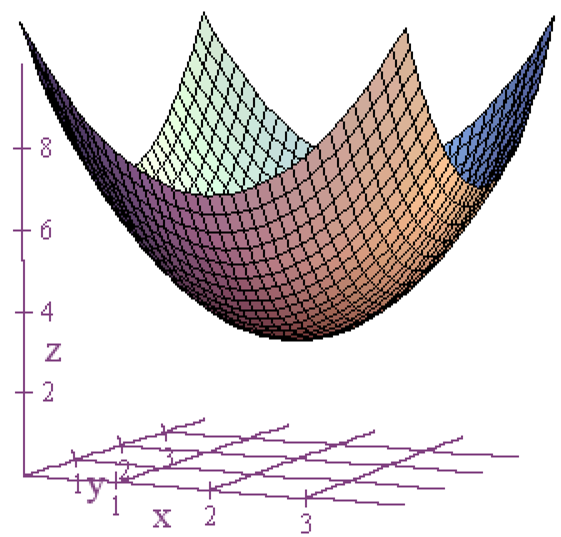


So, suppose we have a surface that is defined parametrically by two parameters.



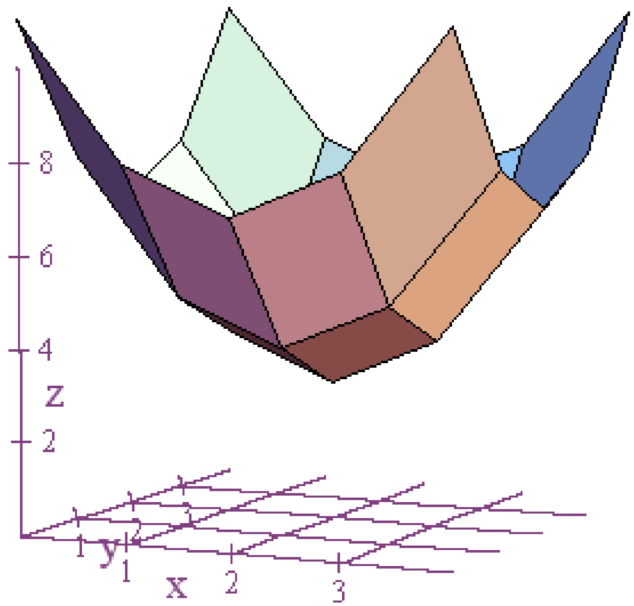
$$\vec{r}(s,t) = x(s,t)\,\hat{i} + y(s,t)\,\hat{j} + z(s,t)\,\hat{k}$$
$$a \le s \le b$$
$$c \le t \le d$$

Then we can develop a formula for surface area in a manner similar to what we did before.



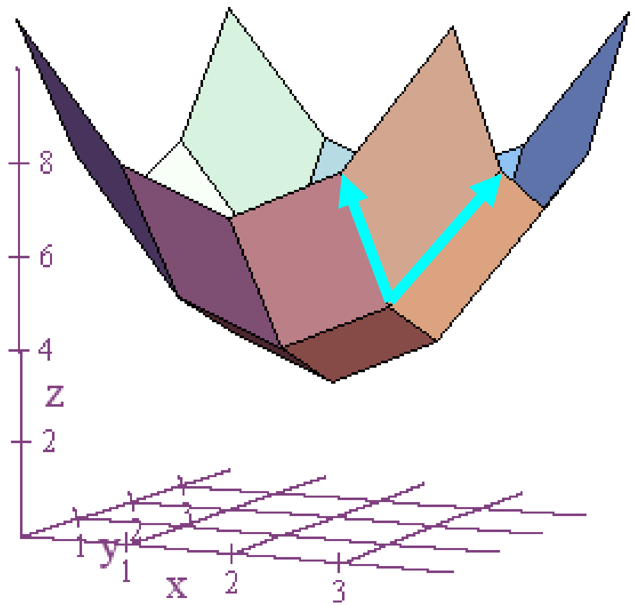
$$\vec{r}(s,t) = x(s,t)\,\hat{i} + y(s,t)\,\hat{j} + z(s,t)\,\hat{k}$$
$$a \le s \le b$$
$$c \le t \le d$$

First, imagine the smooth surface being approximated by a series of parallelograms.



$$\vec{r}(s,t) = x(s,t)\,\hat{i} + y(s,t)\,\hat{j} + z(s,t)\,\hat{k}$$
$$a \le s \le b$$
$$c \le t \le d$$

Each parallelogram on the surface is defined by a pair of vectors, *u* and *v*.

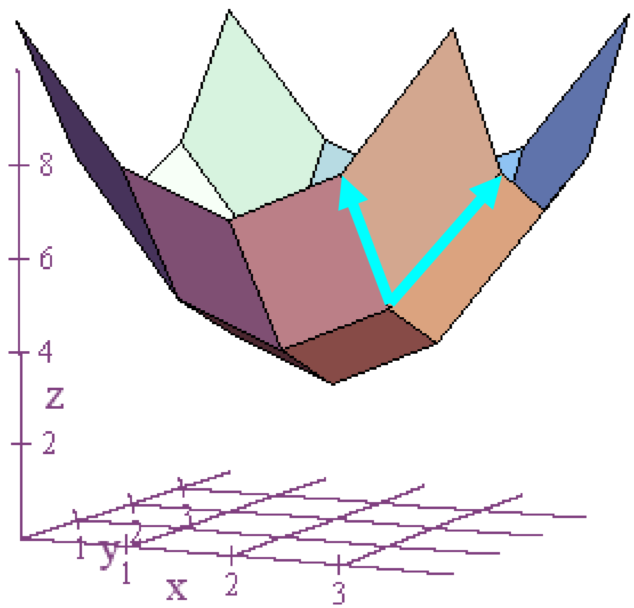


$$\vec{r}(s,t) = x(s,t)\hat{i} + y(s,t)\hat{j} + z(s,t)\hat{k}$$

$$a \le s \le b$$

$$c \le t \le d$$

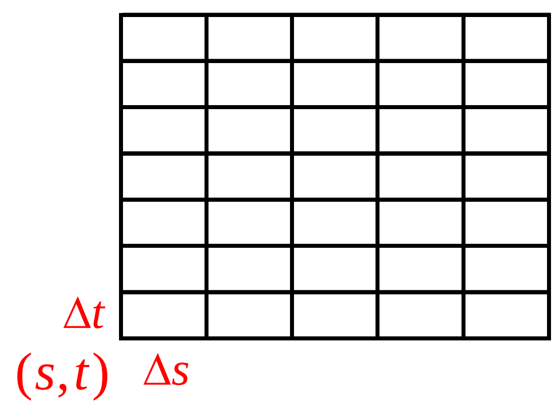
Corresponding to each parallelogram on the surface there will be a rectangle in the *st*-plane.



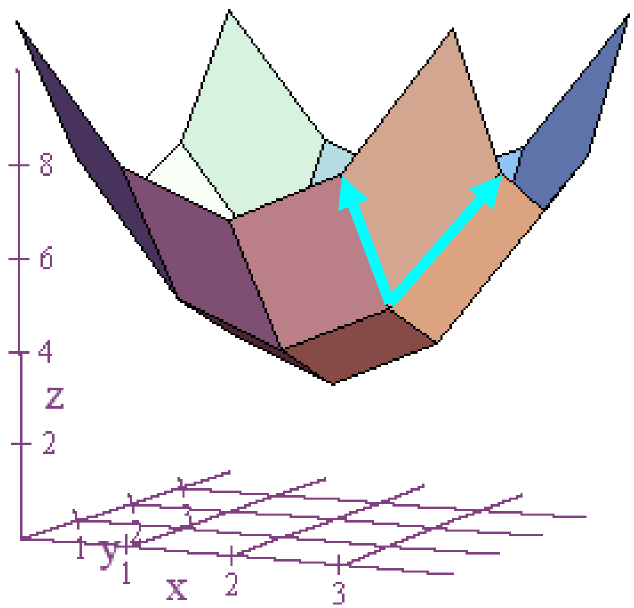
$$\vec{r}(s,t) = x(s,t)\,\hat{i} + y(s,t)\,\hat{j} + z(s,t)\,\hat{k}$$

$$a \le s \le b$$

$$c \le t \le d$$



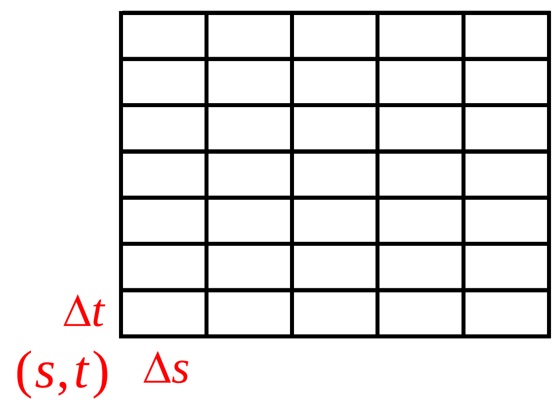
If we designate, on our rectangle in the *st*-plane, the corner point of with the smallest coordinates as (s,t), then the two adjacent corner points will have coordinates (s+Δs,t) and (s,t+Δt).



$$\vec{r}(s,t) = x(s,t)\,\hat{i} + y(s,t)\,\hat{j} + z(s,t)\,\hat{k}$$

$$a \le s \le b$$

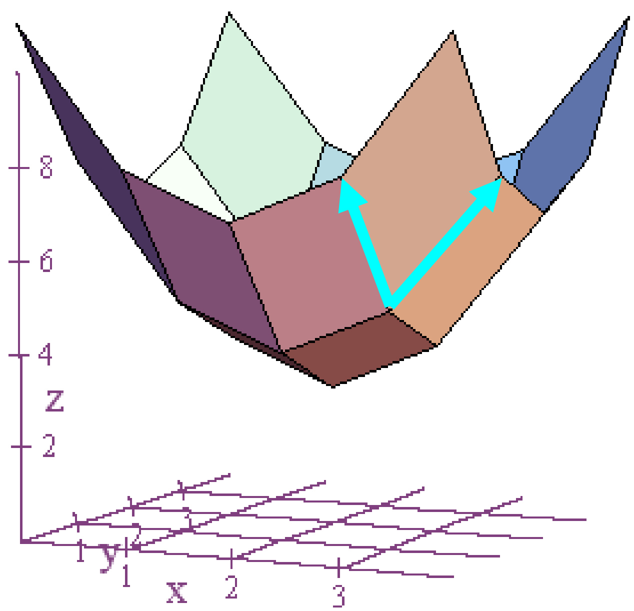
$$c \le t \le d$$



We can now define our vectors *u* and *v* as,

$$\vec{u} = \left[x(s+\Delta s,t) - x(s,t)\right]\hat{i} + \left[y(s+\Delta s,t) - y(s,t)\right]\hat{j} + \left[z(s+\Delta s,t) - z(s,t)\right]\hat{k}$$

$$\vec{v} = \left[x(s,t+\Delta t) - x(s,t)\right]\hat{i} + \left[y(s,t+\Delta t) - y(s,t)\right]\hat{j} + \left[z(s,t+\Delta t) - z(s,t)\right]\hat{k}$$



$$\vec{r}(s,t) = x(s,t)\hat{i} + y(s,t)\hat{j} + z(s,t)\hat{k}$$
$$a \le s \le b$$
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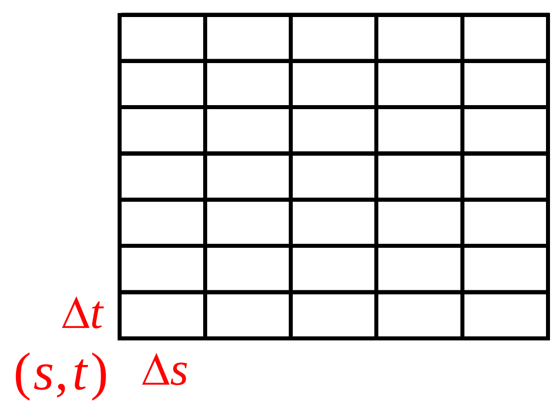
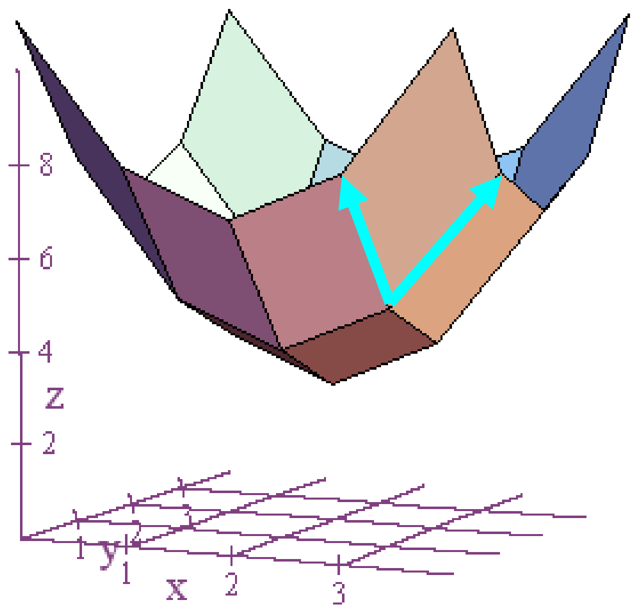


Now think about why this works!

$$\vec{u} \approx \vec{r}_s \Delta s = \frac{\partial x}{\partial s}\Delta s\,\hat{i} + \frac{\partial y}{\partial s}\Delta s\,\hat{j} + \frac{\partial z}{\partial s}\Delta s\,\hat{k}$$

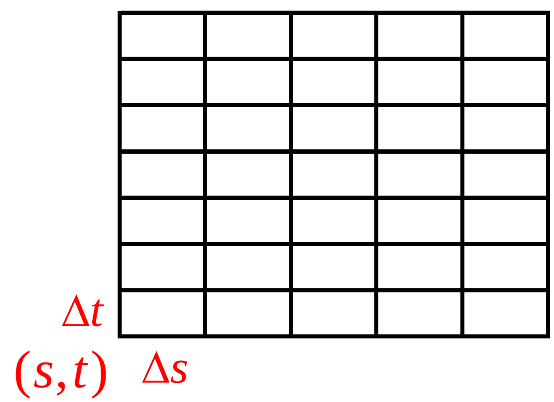
$$\vec{v} \approx \vec{r}_t \Delta t = \frac{\partial x}{\partial t}\Delta t\,\hat{i} + \frac{\partial y}{\partial t}\Delta t\,\hat{j} + \frac{\partial z}{\partial t}\Delta t\,\hat{k}$$



$$\vec{r}(s,t) = x(s,t)\,\hat{i} + y(s,t)\,\hat{j} + z(s,t)\,\hat{k}$$

$$a \le s \le b$$

$$c \le t \le d$$



We can now find the area of a parallelogram using one of the formulas we developed earlier.

$$\vec{u} \approx \vec{r}_s \Delta s = \frac{\partial x}{\partial s}\Delta s\,\hat{i} + \frac{\partial y}{\partial s}\Delta s\,\hat{j} + \frac{\partial z}{\partial s}\Delta s\,\hat{k} \qquad \vec{v} \approx \vec{r}_t \Delta t = \frac{\partial x}{\partial t}\Delta t\,\hat{i} + \frac{\partial y}{\partial t}\Delta t\,\hat{j} + \frac{\partial z}{\partial t}\Delta t\,\hat{k}$$

$$\vec{u}\times\vec{v} \approx \vec{r}_s \Delta s \times \vec{r}_t \Delta t = (\vec{r}_s \times \vec{r}_t)\Delta s \Delta t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} \Delta s \Delta t = \left(\begin{vmatrix} \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} \hat{i} - \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} \hat{j} + \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} \hat{k} \right) \Delta s \Delta t$$

$$= \left(\frac{\partial(y,z)}{\partial(s,t)}\hat{i} - \frac{\partial(x,z)}{\partial(s,t)}\hat{j} + \frac{\partial(x,y)}{\partial(s,t)}\hat{k} \right) \Delta s \Delta t$$

And,

$$\|\vec{u}\times\vec{v}\| \approx \|\vec{r}_s \times \vec{r}_t\| \Delta s \Delta t = \left\| \frac{\partial(y,z)}{\partial(s,t)}\hat{i} - \frac{\partial(x,z)}{\partial(s,t)}\hat{j} + \frac{\partial(x,y)}{\partial(s,t)}\hat{k} \right\| \Delta s \Delta t$$

Therefore,

$$Surface\ Area = \iint_S dS = \lim_{\Delta s, \Delta t \to 0} \sum \left\| \vec{r}_s \times \vec{r}_t \right\| \Delta s \Delta t$$

$$= \iint_T \left\| \vec{r}_s \times \vec{r}_t \right\| ds dt$$

$$dS = \left\| \vec{r}_s \times \vec{r}_t \right\| ds dt$$

<u>Example:</u> Find the surface area of a sphere of radius *r*.

$$Surface\ Area = \iint_S dS = \iint_T \left\| \vec{r}_s \times \vec{r}_t \right\| dsdt$$

$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$

$0 \le \theta \le 2\pi$

$0 \le \varphi \le \pi$

$\rho = r$

$$\vec{r}(\varphi,\theta) = (r\sin\varphi\cos\theta)\hat{i} + (r\sin\varphi\sin\theta)\hat{j} + (r\cos\varphi)\hat{k}$$

$$\vec{r}_\varphi = (r\cos\varphi\cos\theta)\hat{i} + (r\cos\varphi\sin\theta)\hat{j} + (-r\sin\varphi)\hat{k}$$

$$\vec{r}_\theta = (-r\sin\varphi\sin\theta)\hat{i} + (r\sin\varphi\cos\theta)\hat{j}$$

$$Surface\ Area = \iint_S dS = \iint_T \left\| \vec{r}_s \times \vec{r}_t \right\| ds dt$$

$$\vec{r}_\varphi = (r\cos\varphi\cos\theta)\,\hat{i} + (r\cos\varphi\sin\theta)\,\hat{j} + (-r\sin\varphi)\,\hat{k}$$

$$\vec{r}_\theta = (-r\sin\varphi\sin\theta)\,\hat{i} + (r\sin\varphi\cos\theta)\,\hat{j}$$

$$\vec{r}_\varphi \times \vec{r}_\theta = (r^2\sin^2\varphi\cos\theta)\,\hat{i} + (r^2\sin^2\varphi\sin\theta)\,\hat{j} + (r^2\sin\varphi\cos\varphi)\,\hat{k}$$

$$\left\| \vec{r}_\varphi \times \vec{r}_\theta \right\| = \sqrt{r^4\sin^4\varphi\cos^2\theta + r^4\sin^4\varphi\sin^2\theta + r^4\sin^2\varphi\cos^2\varphi}$$

$$= \sqrt{r^4\sin^4\varphi(\cos^2\theta + \sin^2\theta) + r^4\sin^2\varphi\cos^2\varphi} = \sqrt{r^4\sin^4\varphi + r^4\sin^2\varphi\cos^2\varphi}$$

$$= \sqrt{r^4\sin^2\varphi(\sin^2\varphi + \cos^2\varphi)} = \sqrt{r^4\sin^2\varphi} = r^2\sin\varphi$$

$$Surface\ Area = \iint_S dS = \iint_T \left\| \vec{r}_s \times \vec{r}_t \right\| ds dt$$

$$\left\| \vec{r}_\varphi \times \vec{r}_\theta \right\| = r^2 \sin\varphi$$

$$\iint_T r^2 \sin\varphi \, d\varphi d\theta = \int_0^{2\pi} \int_0^{\pi} r^2 \sin\varphi \, d\varphi d\theta = \int_0^{2\pi} -r^2 \cos\varphi \Big|_0^{\pi} d\theta = \int_0^{2\pi} 2r^2 \, d\theta = 2r^2\theta \Big|_0^{2\pi} = 4\pi r^2$$

$$Surface\ Area = \iint_S dS = \iint_T \left\| \vec{r}_s \times \vec{r}_t \right\| ds\,dt$$

$$\left\| \vec{r}_\varphi \times \vec{r}_\theta \right\| = r^2 \sin\varphi$$

$$\iint_T r^2 \sin\varphi \, d\varphi d\theta = \int_0^{2\pi} \int_0^{\pi} r^2 \sin\varphi \, d\varphi d\theta = \int_0^{2\pi} -r^2 \cos\varphi \Big|_0^{\pi} d\theta = \int_0^{2\pi} 2r^2 \, d\theta = 2r^2\theta \Big|_0^{2\pi} = 4\pi r^2$$

$$\text{Surface area of a sphere} = 4\pi r^2$$