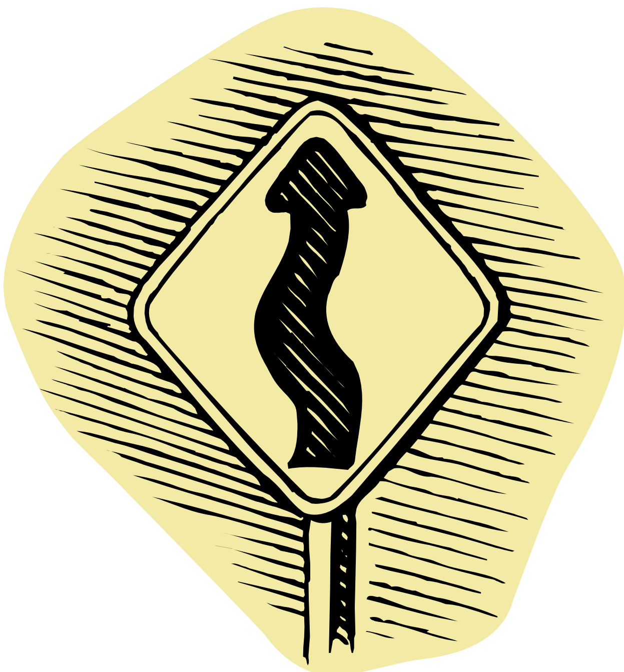
LINE INTEGRALS



If *f* is defined on a smooth curve *C* parametrized by *x=x(t)* and *y=y(t)*, where $a \leq t \leq b$, and if *s* represents arc length, then the line integral of *f* along *C* is:

$$\int_C f(x,y)\,ds = \lim_{\Delta s \to 0} \sum f(x,y)\,\Delta s$$

provided this limit exists.

Using our parametrization:

$$\int_C f(x,y)\,ds = \int_a^b f\big(x(t),y(t)\big)\cdot\frac{ds}{dt}\,dt$$

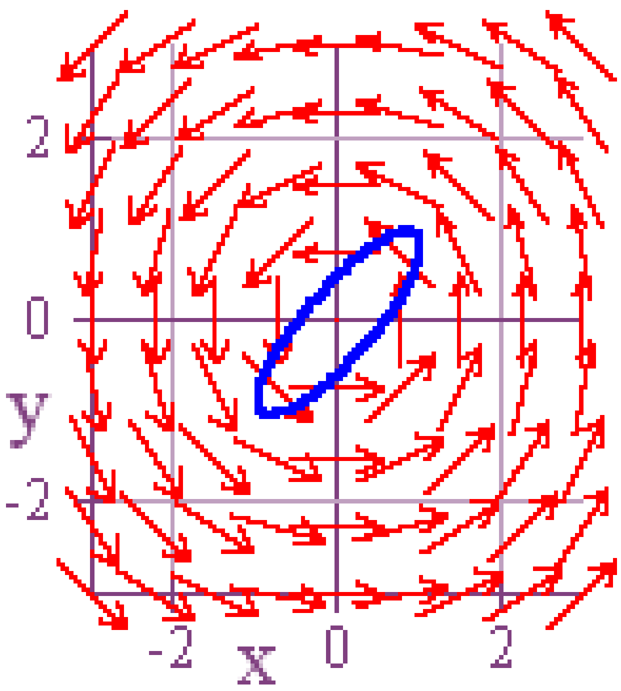
$$= \int_a^b f\big(x(t),y(t)\big)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

We can also integrate along *C* just with respect to either *x* or *y*:

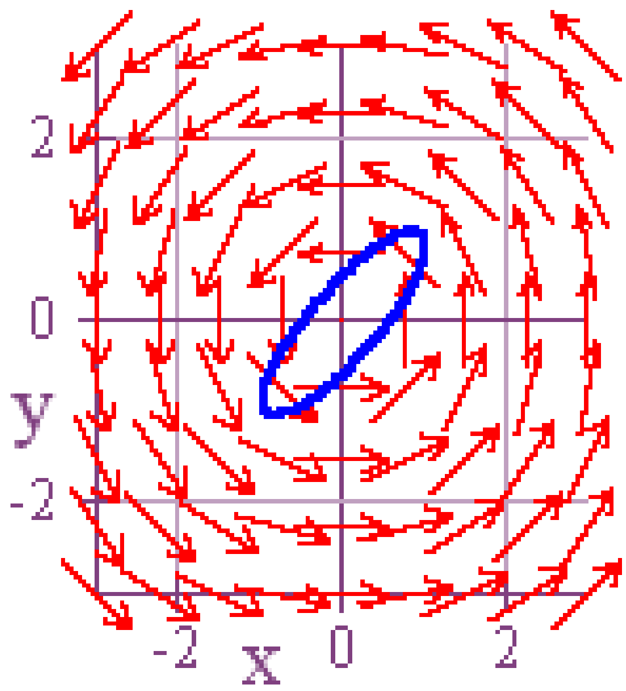
$$\int_C f(x,y)\,dx = \int_a^b f\big(x(t),y(t)\big)\cdot\frac{dx}{dt}\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f\big(x(t),y(t)\big)\cdot\frac{dy}{dt}\,dt$$

A particular application of line integrals is to compute the work done by a force field *F* as it pushes a particle along a path *C*.



Recall that Work = Force x Distance.



Recall also that if our displacement is represented by a vector *D* and the object displaced is acted upon by a force *F* pointing in a different direction, then the work done is equal to the component of *F* in the direction of *D* times the length of *D*. This gives the following:

$$Work = comp_D(F) \bullet \|D\| = \|F\| \cos(\theta) \bullet \|D\| = \|F\| \|D\| \cos(\theta) = F \bullet D$$

If our curve *C* is smooth and if the displacement of our particle is small, then as a result of local linearity, our displacement vector at a point is approximately equal to the change in arc length times the corresponding unit tangent vector. Hence,

$$Work \approx F \bullet (\Delta s \cdot T) = (F \bullet T) \cdot \Delta s$$

If we partition our curve *C* into a series of subintervals of length Δ*s*, then the total work done by the force field in moving the particle along the curve *C* is:

$$Work \approx \sum \left(F \bullet T \right) \cdot \Delta s$$

$$\Rightarrow Work = \lim_{\Delta s \to 0} \sum \left(F \bullet T \right) \cdot \Delta s = \int_C F \bullet T \, ds$$

There are many different ways in which we like to write this last formula:

$$\int_C F \bullet T\, ds = \int_C \left(F \bullet T\right) \frac{ds}{dt}\, dt = \int_C \left(F \bullet T\right) \left\| r'(t) \right\| dt = \int_C \left(F \bullet \frac{r'(t)}{\left\| r'(t) \right\|} \right) \left\| r'(t) \right\| dt$$

$$\int_C \left(F \bullet \frac{r'(t)}{\left\| r'(t) \right\|} \cdot \left\| r'(t) \right\| \right) dt = \int_C \left(F \bullet r'(t) \right) dt = \int_C \left(F \bullet \frac{dr}{dt} \right) dt = \int_C F \bullet dr$$

If *F=<P,Q> and r=<x(t),y(t)>, then we get the following*:

$$F(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$$

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}, \qquad a \le t \le b$$

$$\int_C F \bullet T\, ds = \int_C F \bullet dr = \int_C \left(F \bullet \frac{dr}{dt} \right) dt$$

$$= \int_a^b \left(P\hat{i} + Q\hat{j} \right) \bullet \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} \right) dt = \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} \right) dt = \int_C Pdx + Qdy$$

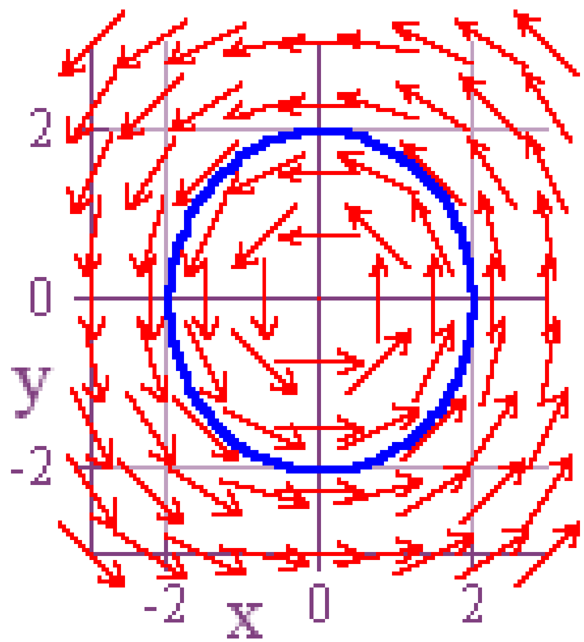
Hence,

$$F(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$$

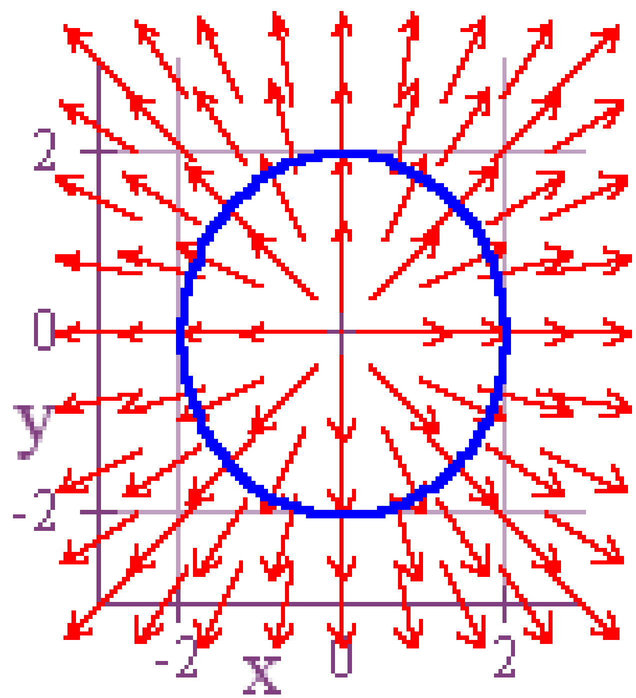
$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}, \qquad a \le t \le b$$

$$\int_C F \bullet T \, ds = \int_C F \bullet dr = \int_C P dx + Q dy$$

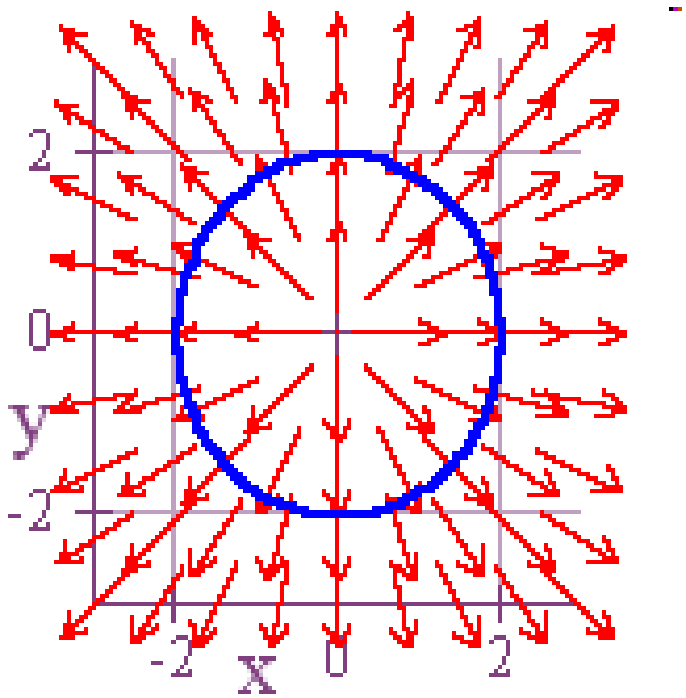
Notice that the more the vectors in a force field tend to point in the direction of a closed curve *C*, the more the force field will tend to generate circulation of a point along the curve *C*.



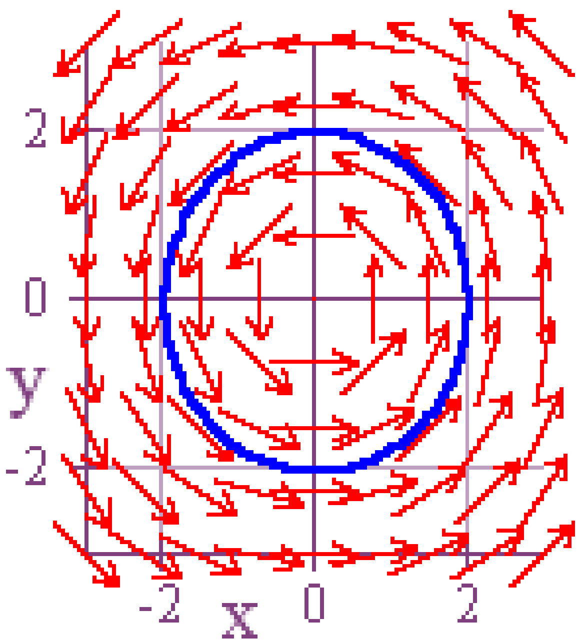
On the other hand, if you take a vector from the force field below and look at its component in the direction of a unit tangent vector at a point on the closed curve *C*, then that component is equal to zero.



Thus, this force field produces no circulation of a point along the closed curve *C*.



The bottom line is that the same integral that computes work done by a force field in moving a point along a closed curve also measures the measures the tendency for circulation to be generated along that curve.



$$Circulation = \int_C F \bullet T\, ds = \int_C F \bullet dr$$
$$= Work$$

Examples:

$$C: x=\cos(t),\ y=\sin(t),\ \ 0\le t\le 2\pi$$

$$\int_C y^2 x\,ds = \int_0^{2\pi} \sin^2(t)\cos(t)\sqrt{\left(-\sin(t)\right)^2+\left(\cos(t)\right)^2}\,dt$$

$$= \int_0^{2\pi} \sin^2(t)\cos(t)\,dt = \int_0^0 u^2\,du = 0$$

Examples:

$$C: x=\cos(t),\ y=\sin(t),\ \ 0\le t\le 2\pi$$

$$F=-y\hat{i}+x\hat{j}$$

$$Work=\int_C F\bullet dr=\int_C Pdx+Qdy=\int_a^b\left(-y\frac{dx}{dt}+x\frac{dy}{dt}\right)dt$$

$$=\int_0^{2\pi}\Big(-\sin(t)\big(-\sin(t)\big)+\cos(t)\cos(t)\Big)dt=\int_0^{2\pi}1dt=t\Big|_0^{2\pi}=2\pi$$

Examples:

$$C: x = \cos(t),\ y = \sin(t),\ z = t,\quad 0 \le t \le 2\pi$$

$$\int_C y\sin(z)\,ds = \int_0^{2\pi} (\sin t)\sin t\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \int_0^{2\pi} \sin^2 t\sqrt{\sin^2 t + \cos^2 t + 1}\,dt = \sqrt{2}\int_0^{2\pi} \sin^2 t\,dt = \sqrt{2}\int_0^{2\pi} \frac{1-\cos 2t}{2}\,dt$$

$$= \frac{\sqrt{2}}{2}\left(t - \frac{\sin(2t)}{2}\right)\Bigg|_0^{2\pi} = \pi\sqrt{2}$$