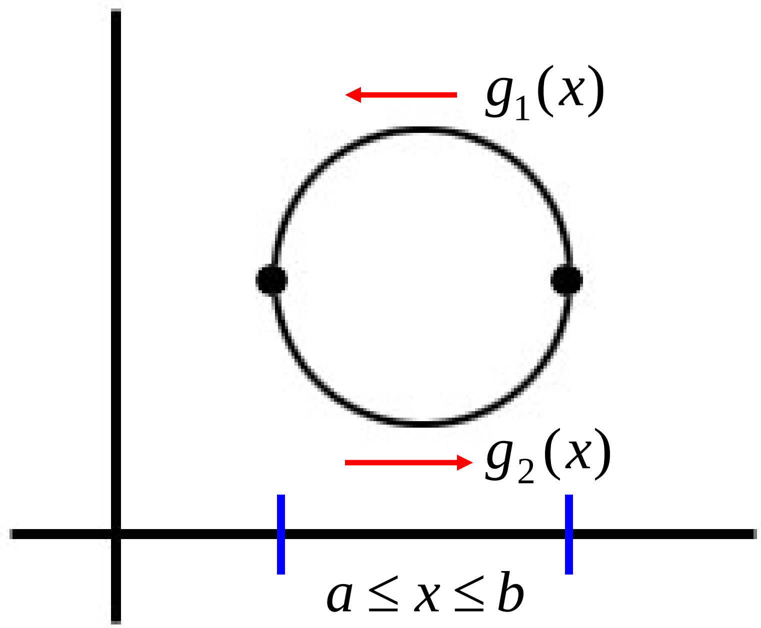
GREEN’S THEOREM

Theorem: Let *C* be a smooth, simple closed curve in the plane that is oriented counter-clockwise, and let *R* be the region bounded by *C*. If *P* and *Q* have continuous partial derivatives on an open region that contains *R*, then,

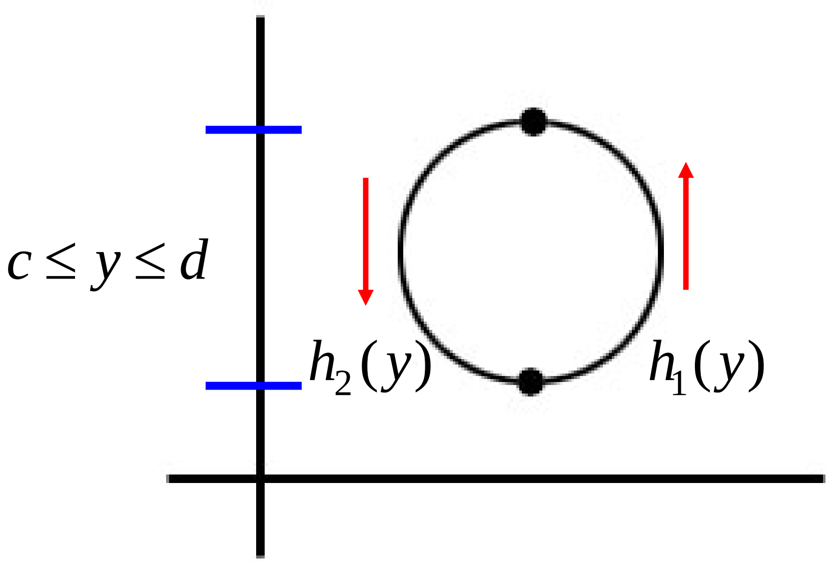
$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Proof: We'll do just a special case. Thus, suppose our curve *C* and region *R* look something like the following:

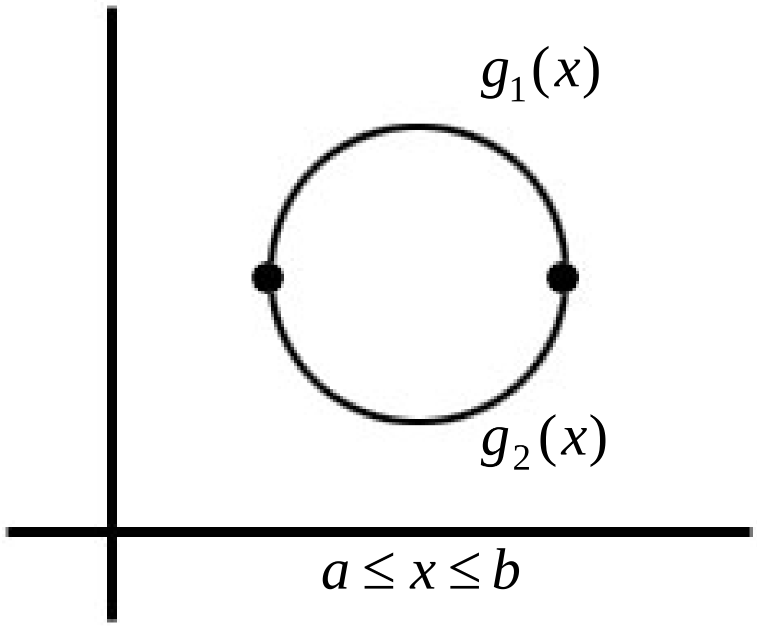


In this case, we can break the curve into a top part and a bottom part over an interval on the x-axis from *a* to *b*.

Or, we could just as easily portray *x* as varying from *h2* to *h1* as *y* varies from *c* to *d*.



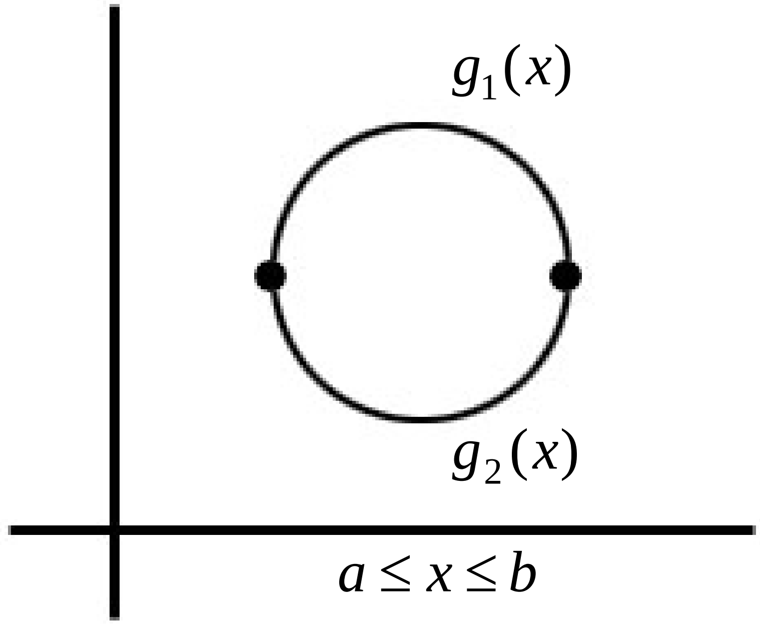
Now let's begin. Suppose the curve below is oriented in the counterclockwise direction and is parametrized by *x*. Then,



$$\int_C P\,dx = \int_a^b P(x, g_2(x))\,dx + \int_b^a P(x, g_1(x))\,dx$$

$$= \int_a^b P(x, g_2(x))\,dx - \int_a^b P(x, g_1(x))\,dx = \int_a^b \left[P(x, g_2(x)) - P(x, g_1(x))\right] dx$$

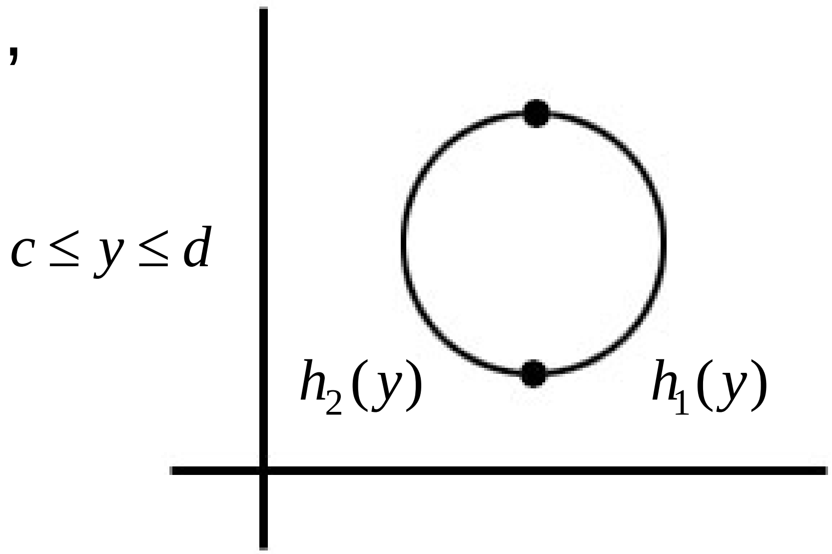
Similarly,



$$-\iint_R \frac{\partial P}{\partial y}\, dA = -\int_a^b \int_{g_2(x)}^{g_1(x)} \frac{\partial P}{\partial y}\, dy dx = -\int_a^b \left[P(x, g_1(x)) - P(x, g_2(x))\right] dx$$

$$= \int_a^b \left[P(x, g_2(x)) - P(x, g_1(x))\right] dx = \int_C P\, dx$$

Likewise,

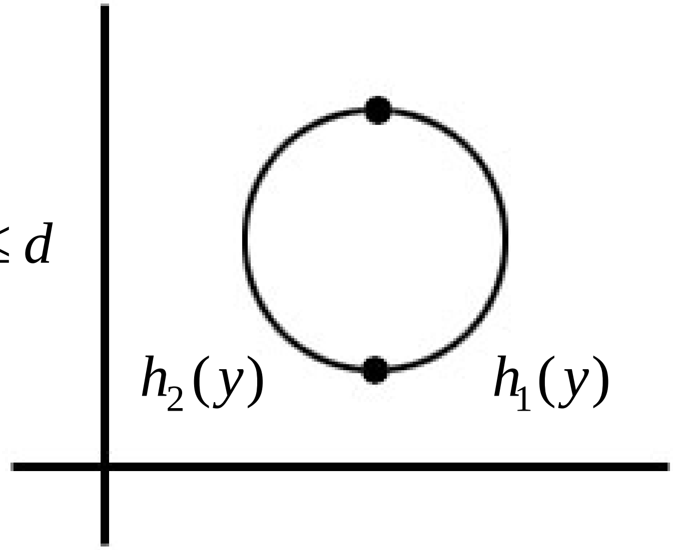


$$\iint_R \frac{\partial Q}{\partial x}\, dA = \int_c^d \int_{h_2(y)}^{h_1(y)} \frac{\partial Q}{\partial x}\, dx dy = \int_c^d \left[Q(h_1(y), y) - Q(h_2(y), y) \right] dy$$

$$= \int_c^d Q(h_1(y), y)\, dy + \int_d^c Q(h_2(y), y)\, dy = \int_C Q\, dy$$

Likewise,

$$c \le y \le d$$



$$\iint_R \frac{\partial Q}{\partial x} dA = \int_c^d \int_{h_2(y)}^{h_1(y)} \frac{\partial Q}{\partial x} dx dy = \int_c^d \left[Q(h_1(y), y) - Q(h_2(y), y) \right] dy$$

$$= \int_c^d Q(h_1(y), y)\, dy + \int_d^c Q(h_2(y), y)\, dy = \int_C Q\, dy$$

Therefore, $$\int_C P\, dx + Q\, dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

EXAMPLE:

Evaluate $\int_C x^4\,dx + xy\,dy$ where C is the triangle defined by the line segments connecting $(0,0)$ to $(1,0)$, $(1,0)$ to $(0,1)$, and $(0,1)$ to $(0,0)$.

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$$\int_C x^4\,dx + xy\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_0^1 \int_0^{1-x} (y-0)\,dy\,dx$$

$$= \int_0^1 \frac{y^2}{2}\bigg|_0^{1-x} dx = \int_0^1 \frac{(1-x)^2}{2}\,dx = \frac{-(1-x)^3}{2\cdot 3}\bigg|_0^1 = \frac{1}{6}$$

EXAMPLE:

Evaluate $\int_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$ where C is the circle $x^2 + y^2 = 9$.

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$$\int_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$$

$$= \iint_R \left[\frac{\partial(7x + \sqrt{y^4+1})}{\partial x} - \frac{\partial(3y - e^{\sin x})}{\partial y} \right] dA$$

$$= \iint_R (7-3)\,dA = \int_0^{2\pi} \int_0^3 4r\,dr\,d\theta = \int_0^{2\pi} 2r^2 \Big|_0^3 \,d\theta$$

$$= \int_0^{2\pi} 18\,d\theta = 18\theta\Big|_0^{2\pi} = 36\pi$$