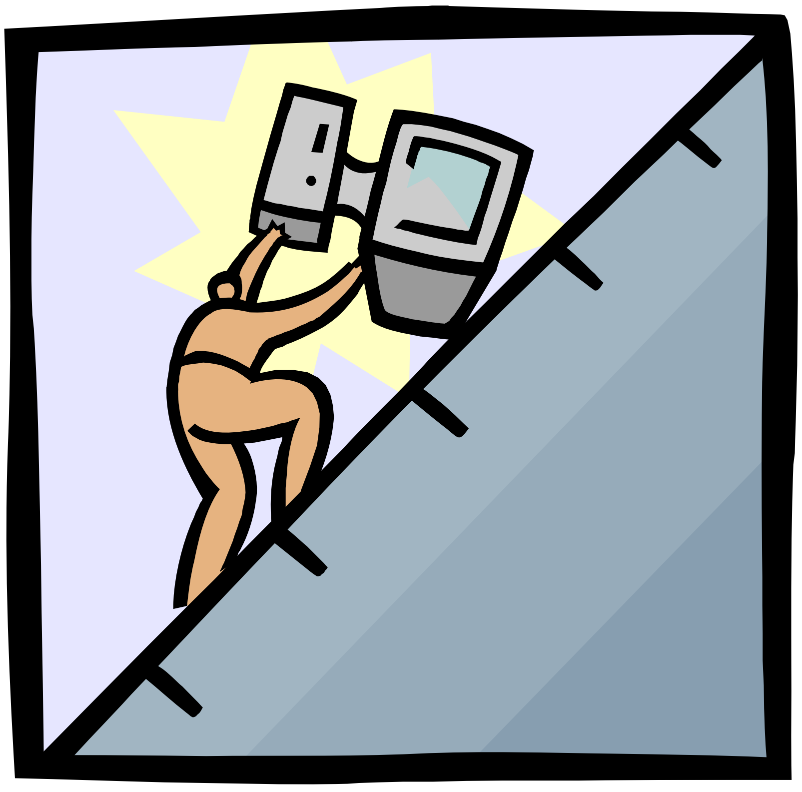
The Gradient and Level Curves



Definition: A parametrized curve *r(t)* is called smooth if *r´(t)* is continuous and *r´(t)* is never the zero vector (except possibly at the endpoints).

<u>Theorem:</u> Let *z=f(x,y)* be differentiable at *(a,b)* and let *f(a,b)=c*. Also, let *C* be the level curve *f(x,y)=c* that passes through *(a,b)*. If *C* is smooth with smooth paramtrization r(t) and if *grad f(a,b)* is not equal to 0, then *grad f(a,b)* is normal to *C* at *(a,b)*.

In other words, *grad f* is perpendicular to *r´(t)* at *(a,b)*.

NOTE: Another notation for the gradient of f is ∇f.

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$$Therefore, \nabla f\ is\ normal\ to\ C\ at\ (a,b).$$

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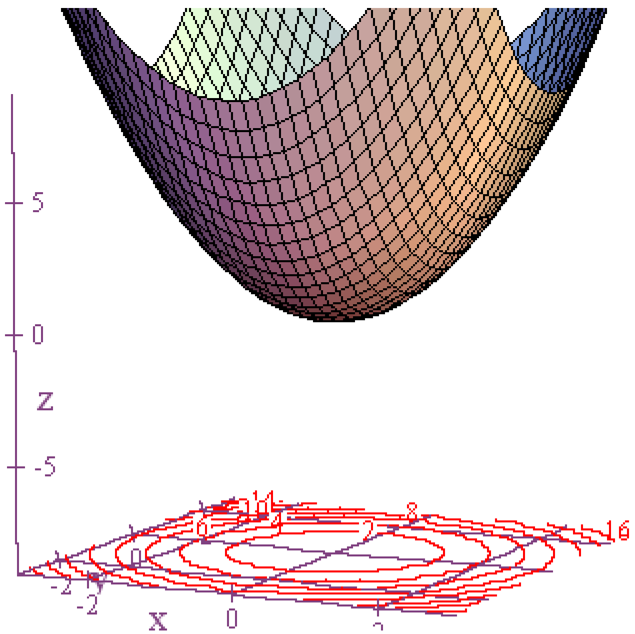

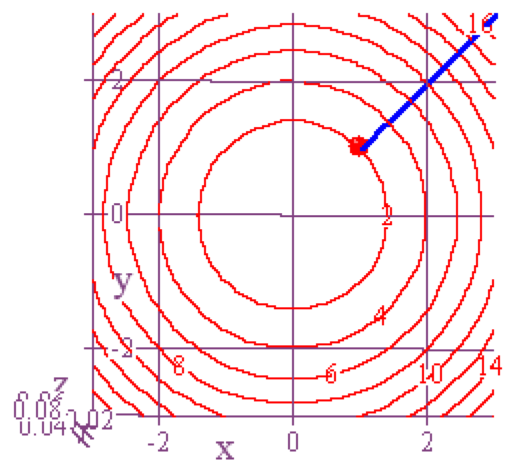

A similar proof can be constructed to show that if $w = f(x, y, z)$, then the gradient vector $\nabla w = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$ evaluated at (a,b,c) is normal to the level surface $w = f(a,b,c)$.

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Also, if this gradient vector is normal to the level surface $0 = x^2 + y^2 - z$, at the point $P = (1,1,2)$, then we should be able to use this information to find the tangent plane at this point.

$$z = f(x, y) = x^2 + y^2$$

$$P = (1, 1, 2)$$

$$0 = x^2 + y^2 - z$$

$$w = x^2 + y^2 - z$$

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$$Q = (x,y,z)$$

$$\overrightarrow{PQ} = (x-1)\hat{i} + (y-1)\hat{j} + (z-2)\hat{k}$$

$$\nabla w(1,1,2) \bullet \overrightarrow{PQ} = 0$$

$$2(x-1) + 2(y-1) - 1(z-2) = 0$$

$$\Rightarrow 2x + 2y - z - 2 = 0$$

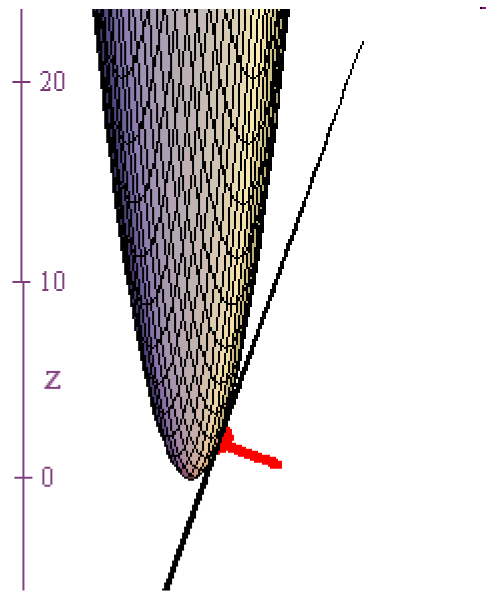
$$\Rightarrow z = 2x + 2y - 2$$

$$z = f(x, y) = x^2 + y^2$$

$$P = (1, 1, 2)$$

$$0 = x^2 + y^2 - z$$

$$w = x^2 + y^2 - z$$



$$\nabla w = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\nabla w(1,1,2) = 2\hat{i} + 2\hat{j} - \hat{k}$$

$$2(x-1) + 2(y-1) - 1(z-2) = 0$$

$$\Rightarrow 2x + 2y - z - 2 = 0$$

$$\Rightarrow z = 2x + 2y - 2$$

Notice that

$$2(x-1)+2(y-1)-1(z-2)=0$$

$$\Rightarrow \frac{\partial f(a,b)}{\partial x}(x-a)+\frac{\partial f(a,b)}{\partial y}(y-b)-(z-f(a,b))=0$$

$$\Rightarrow z=\frac{\partial f(a,b)}{\partial x}(x-a)+\frac{\partial f(a,b)}{\partial y}(y-b)+f(a,b)$$

is the same result we found previously for the equation of a tangent plane.