# FUBINI'S THEOREM

**Fubini's Theorem:** If *f* is continuous on the rectangle, $R = \left\{ (x, y) \middle| a \le x \le b, c \le y \le d \right\}$

then,

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

## Arguement:

$$\iint_R f(x,y)\,dA \approx \sum_{i,j} f(x,y)\,\Delta A = \sum_i \left( \sum_j f(x,y)\,\Delta y \right) \Delta x = \sum_j \left( \sum_i f(x,y)\,\Delta x \right) \Delta y$$

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$$\iint_R f(x,y)\,dA = \lim_{\Delta x,\Delta y \to 0} \sum f(x,y)\,\Delta A = \lim_{\Delta x \to 0} \sum_i \left( \lim_{\Delta y \to 0} \sum_j f(x,y)\,\Delta y \right) \Delta x = \int_a^b \left( \int_c^d f(x,y)\,dy \right) dx$$

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## Example 1:

$$\int_0^1 \int_2^4 xy^2 \, dy dx = \int_0^1 \left( \frac{xy^3}{3} \right) \Bigg|_2^4 dx = \int_0^1 \left( \frac{64x}{3} - \frac{8x}{3} \right) dx = \int_0^1 \frac{56x}{3} dx$$

$$= \frac{28x^2}{3} \Bigg|_0^1 = \frac{28}{3} - \frac{0}{3} = \frac{28}{3}$$

# Example 1:

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$$= \frac{28x^2}{3}\Bigg|_0^1 = \frac{28}{3} - \frac{0}{3} = \frac{28}{3}$$

$$\int_2^4\int_0^1 xy^2\,dx\,dy = \int_2^4 \left(\frac{x^2y^2}{2}\right)\Bigg|_0^1 dy = \int_2^4 \left(\frac{y^2}{2} - \frac{0}{2}\right) dy = \int_2^4 \frac{y^2}{2}\,dy$$

$$= \frac{y^3}{6}\Bigg|_2^4 = \frac{64}{6} - \frac{8}{6} = \frac{56}{6} = \frac{28}{3}$$

## Example 2:

Use a double integral to find the area of the region between the curves $y = x^2$ and $y = x^3$ from $x = 0$ to $x = 1$.

$z = f(x, y) = 1$

$0 \le x \le 1$

$x^3 \le y \le x^2$

$$\text{Area } = \iint_R dA = \int_0^1 \int_{x^3}^{x^2} dy dx = \int_0^1 y \Big|_{x^3}^{x^2} dx = \int_0^1 (x^2 - x^3) dx$$

$$= \left( \frac{x^3}{3} - \frac{x^4}{4} \right) \Bigg|_0^1 = \left( \frac{1}{3} - \frac{1}{4} \right) - \left( \frac{0}{3} - \frac{0}{4} \right) = \frac{4}{12} - \frac{3}{12} = \frac{1}{12}$$

# Example 2:

Use a double integral to find the area of the region between the curves $y = x^2$ and $y = x^3$ from $x = 0$ to $x = 1$.

$z = f(x, y) = 1$

$0 \le y \le 1$

$\sqrt{y} \le x \le \sqrt[3]{y}$

$$\text{Area} = \iint_R dA = \int_0^1 \int_{\sqrt{y}}^{\sqrt[3]{y}} dx\,dy = \int_0^1 x\Big|_{\sqrt{y}}^{\sqrt[3]{y}} dy = \int_0^1 (y^{1/3} - y^{1/2})\,dy$$

$$= \left(\frac{3y^{4/3}}{4} - \frac{2y^{3/2}}{3}\right)\Bigg|_0^1 = \left(\frac{3}{4} - \frac{2}{3}\right) - \left(\frac{0}{4} - \frac{0}{3}\right) = \frac{9}{12} - \frac{8}{12} = \frac{1}{12}$$

**Example 3:** Evaluate $\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\,dydx.$

$0 \le x \le 6$

$\frac{x}{3} \le y \le 2$

$0 \le y \le 2$

$0 \le x \le 3y$

$$\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\,dydx = \int_0^2 \int_0^{3y} x\sqrt{y^3+1}\,dxdy = \int_0^2 \left. \frac{x^2\sqrt{y^3+1}}{2} \right|_0^{3y} dy$$

$$= \int_0^2 \frac{9y^2\sqrt{y^3+1}}{2}\,dy = \int_1^9 \frac{3u^{1/2}}{2}\,du = \left. u^{3/2} \right|_1^9 = 9^{3/2} - 1^{3/2} = 27 - 1 = 26$$