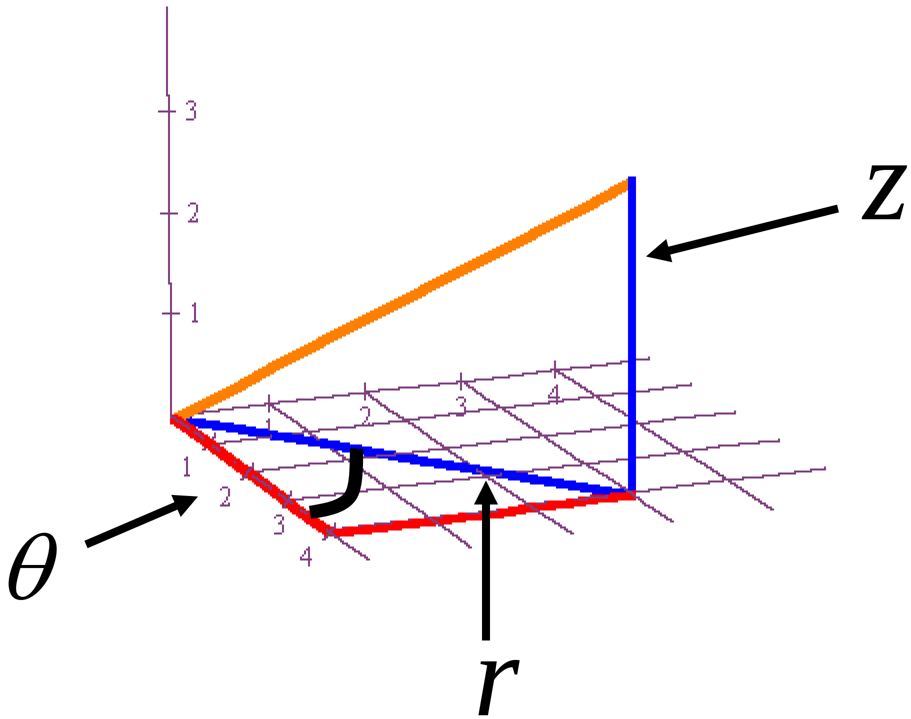
INTEGRALS IN POLAR AND CYLINDRICAL COORDINATES

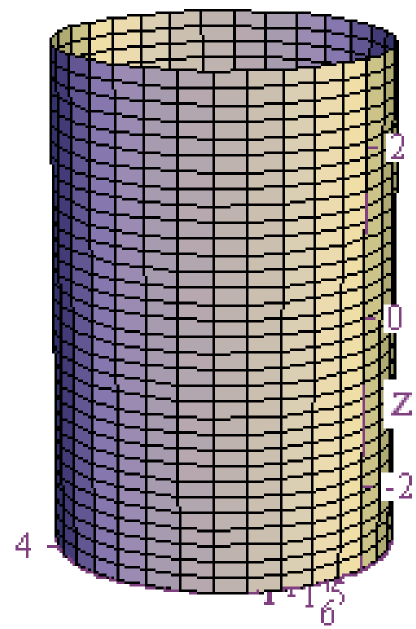
$$(r, \theta, z)$$



$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$z = z$$

$$r^2 = x^2 + y^2$$
$$\tan\theta = \frac{y}{x}$$

$$0 \le \theta \le 2\pi$$
$$0 \le r < \infty$$
$$-\infty < z < \infty$$

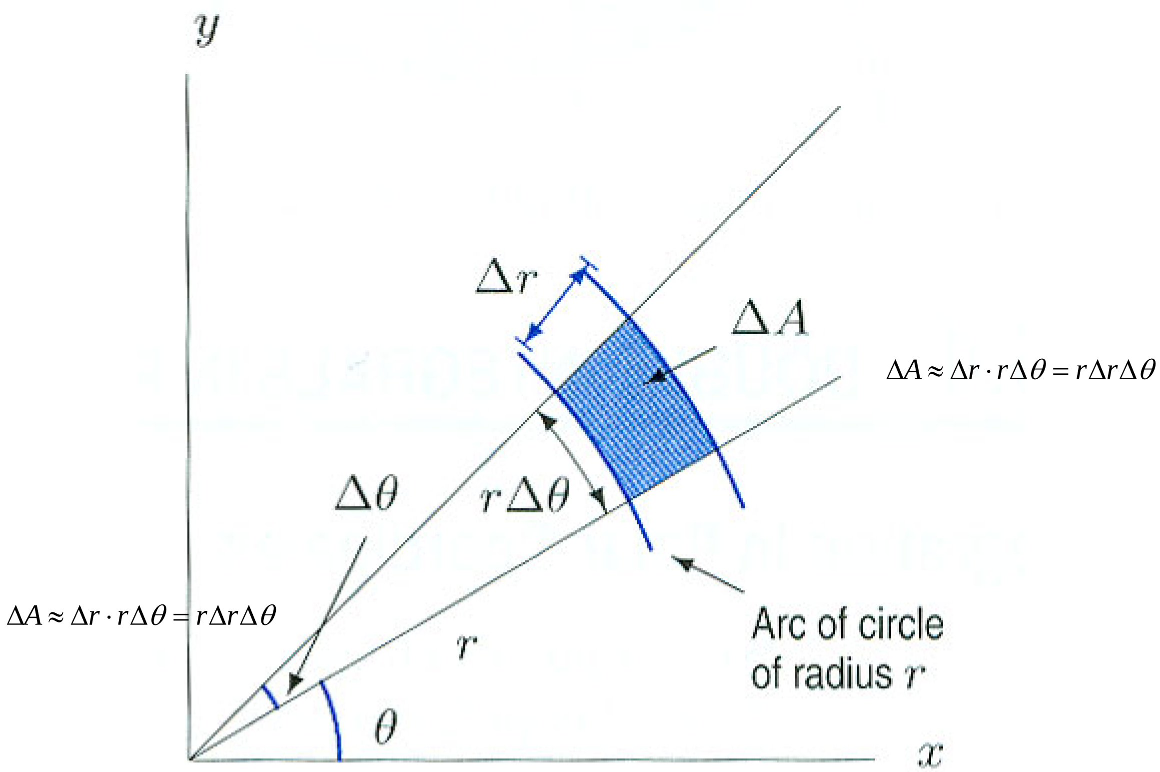


$$r = 2$$

$$0 \le \theta \le 2\pi$$

$$-3 \le z \le 3$$

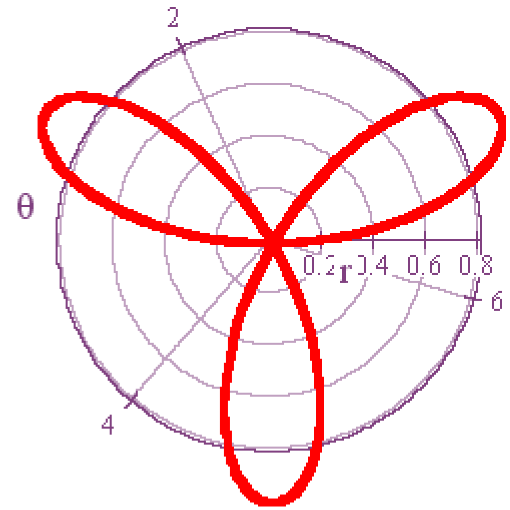
DOUBLE INTEGRALS IN POLAR COORDINATES



$$\Delta A \approx \Delta r \cdot r\Delta\theta = r\Delta r\Delta\theta$$

$$dA = r\,dr\,d\theta$$

EXAMPLE: Find the area of one petal of the three petal rose that is the graph of $r = \sin(3\theta)$.



Let,

$z = 1$

$0 \le \theta \le \frac{\pi}{3}$

$0 \le r \le \sin(3\theta)$

$$\text{Area} = \iint_R dA = \int_0^{\pi/3} \int_0^{\sin(3\theta)} r\,dr\,d\theta = \int_0^{\pi/3} \left.\frac{r^2}{2}\right|_0^{\sin(3\theta)} d\theta$$

$$= \int_0^{\pi/3} \frac{\sin^2(3\theta)}{2} d\theta = \frac{1}{6}\int_0^{\pi} \sin^2 u\,du = \frac{1}{6}\int_0^{\pi} \frac{1-\cos 2u}{2} du$$

$$= \frac{1}{6}\left.\left(\frac{u}{2} - \frac{\sin 2u}{4}\right)\right|_0^{\pi} = \frac{\pi}{12}$$

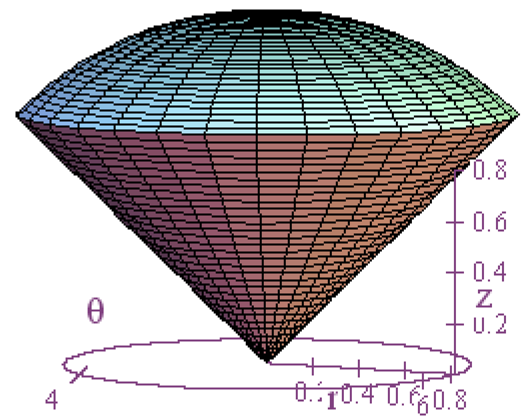
EXAMPLE: Find the integral of $z = \dfrac{1}{(x^2+y^2)^{3/2}}$ on the region R corresponding to $0 \le \theta \le \pi/4$ and $1 \le r \le 2$.

$$\iint_R \frac{1}{(x^2+y^2)^{3/2}}\,dA = \int_0^{\pi/4}\int_1^2 \frac{1}{(r^2)^{3/2}}\,r\,dr\,d\theta = \int_0^{\pi/4}\int_1^2 r^{-2}\,dr\,d\theta$$

$$= \int_0^{\pi/4} -\frac{1}{r}\Big|_1^2\,d\theta = \int_0^{\pi/4}\frac{1}{2}\,d\theta = \frac{\theta}{2}\Big|_0^{\pi/4} = \frac{\pi}{8}$$

EXAMPLE: Find the volume of the ice cream cone defined by

$-1 \le x \le 1,\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2},\ \sqrt{x^2+y^2} \le z \le \sqrt{2-(x^2+y^2)}.$



$$\text{Volume } = \iiint_R dV$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-(x^2+y^2)}} dz\,dy\,dx$$

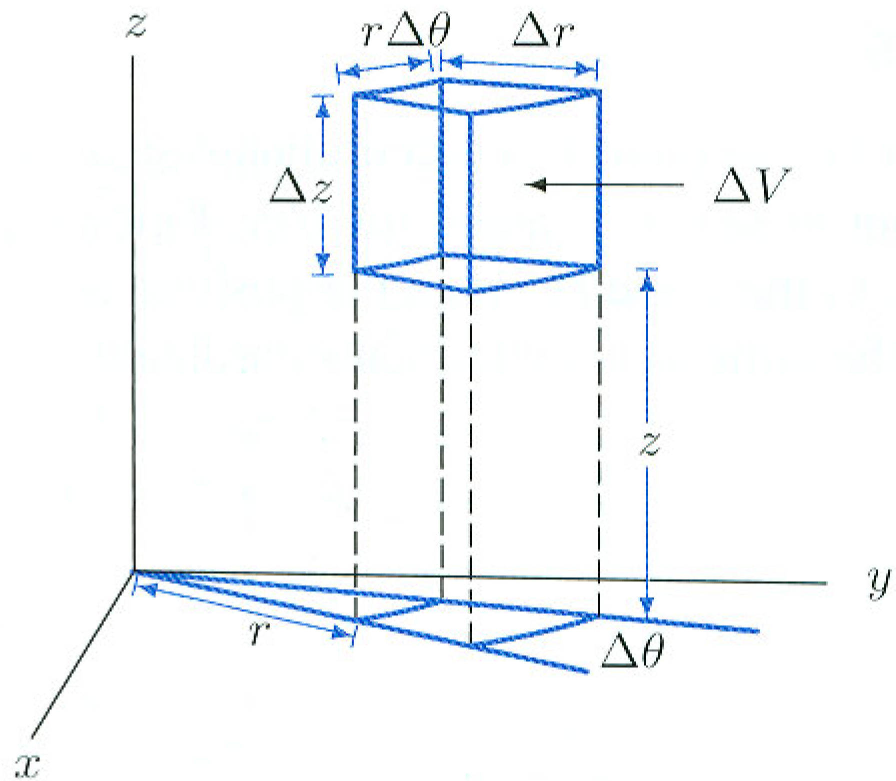
$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\sqrt{2-(x^2+y^2)} - \sqrt{x^2+y^2} \right) dy\,dx$$

$$= \int_0^{2\pi} \int_0^1 \left(\sqrt{2-r^2} - r \right) r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left(r\sqrt{2-r^2} - r^2 \right) dr\, d\theta = \int_0^{2\pi} \left(\frac{-(2-r^2)^{3/2}}{3} - \frac{r^3}{3} \right) \Bigg|_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{(2^{3/2}-2)}{3} d\theta = \frac{\theta(2^{3/2}-2)}{3} \Bigg|_0^{2\pi} = \frac{2\pi(2^{3/2}-2)}{3} = \frac{4\pi}{3}\left(\sqrt{2}-1\right)$$

TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES

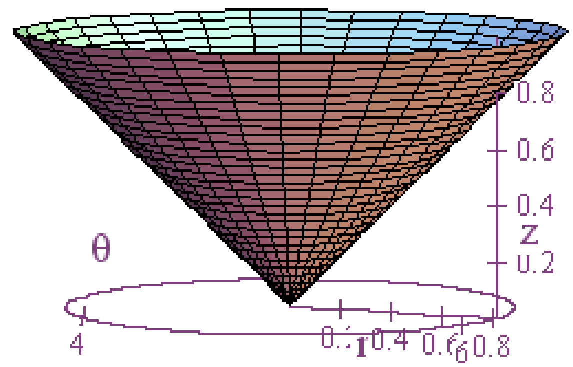


$$\Delta V \approx r\Delta\theta\Delta r\Delta z = r\Delta r\Delta\theta\Delta z$$

$$dV = r\,dr\,d\theta\,dz$$

EXAMPLE: Find the volume beneath the cone defined by

$$z = \sqrt{x^2 + y^2} = r,\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi.$$

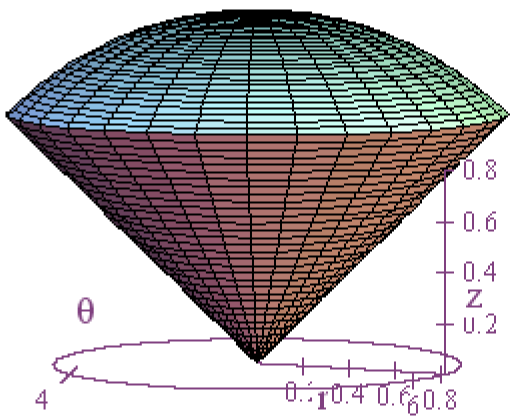


$$\text{Volume } = \iiint_R dV = \int_0^{2\pi} \int_0^1 \int_0^r r\, dz dr d\theta = \int_0^{2\pi} \int_0^1 rz\Big|_0^r dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 r^2\, dr d\theta = \int_0^{2\pi} \frac{r^3}{3}\Big|_0^1 d\theta = \int_0^{2\pi} \frac{1}{3} d\theta = \frac{\theta}{3}\Big|_0^{2\pi} = \frac{2\pi}{3}.$$

EXAMPLE: Find the volume of the ice cream cone defined by

$r \le z \le \sqrt{2-r^2}$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$.



$$\text{Volume } = \iiint_R dV = \int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r\,dzdrd\theta = \int_0^{2\pi}\int_0^1 rz\Big|_r^{\sqrt{2-r^2}}\,drd\theta$$

$$= \int_0^{2\pi}\int_0^1\left(r\sqrt{2-r^2}-r^2\right)drd\theta = \int_0^{2\pi}\left(\frac{-(2-r^2)^{3/2}}{3}-\frac{r^3}{3}\right)\Bigg|_0^1 d\theta$$

$$= \int_0^{2\pi}\frac{(2^{3/2}-2)}{3}d\theta = \frac{\theta(2^{3/2}-2)}{3}\Bigg|_0^{2\pi} = \frac{2\pi(2^{3/2}-2)}{3} = \frac{4\pi}{3}\left(\sqrt{2}-1\right)$$