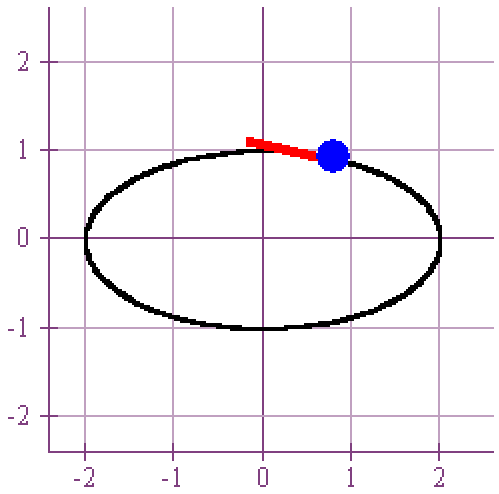
CURVATURE

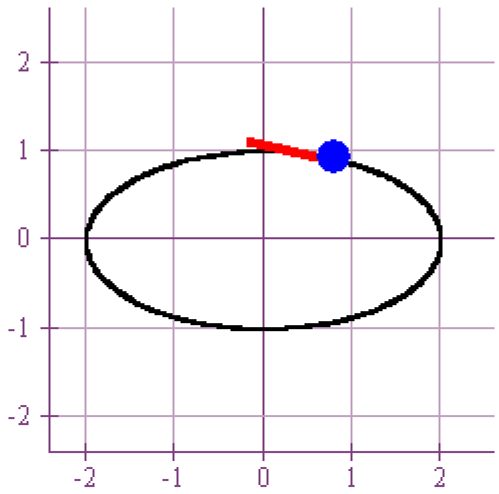
We define curvature as the magnitude of the rate of change of the unit tangent vector with respect to arc length.

$$curvature = \kappa = \left\| \frac{dT}{ds} \right\|$$



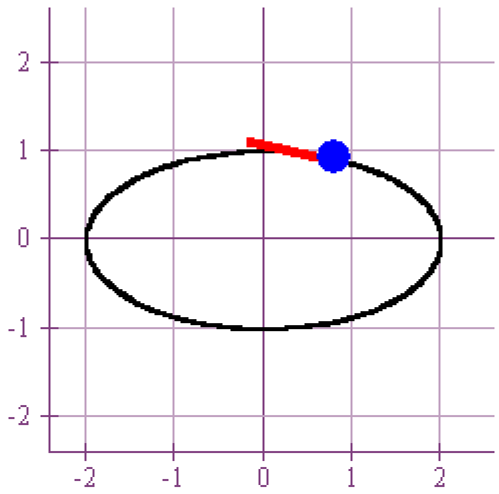
Why does this definition make sense?

$$curvature = \kappa = \left\| \frac{dT}{ds} \right\|$$



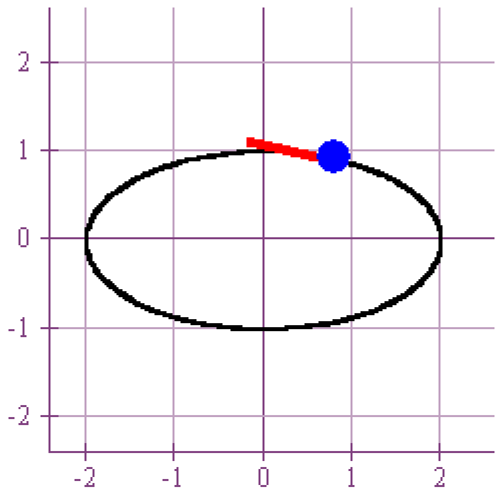
Simply because the length of the unit tangent isn't going to change. The only way you'll get a lot of change is if the direction of the vector changes quickly.

$$curvature = \kappa = \left\| \frac{dT}{ds} \right\|$$



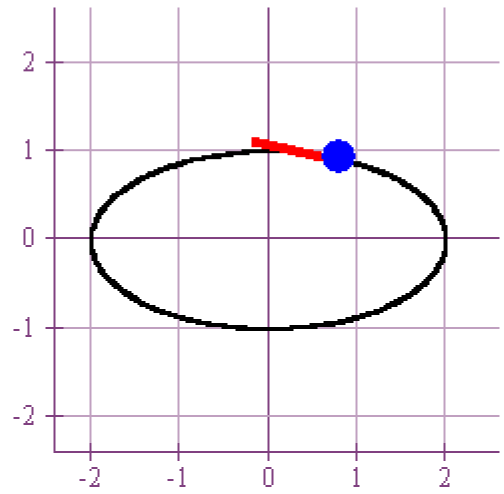
In other words, if there is a lot of curvature.

$$curvature = \kappa = \left\| \frac{dT}{ds} \right\|$$



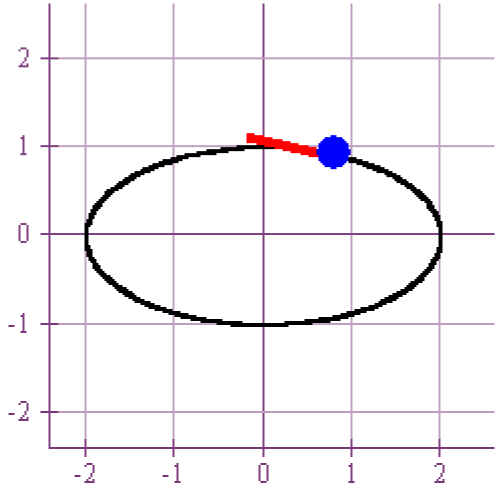
In evaluating curvature, though, there's just one big problem.

$$curvature = \kappa = \left\| \frac{dT}{ds} \right\|$$



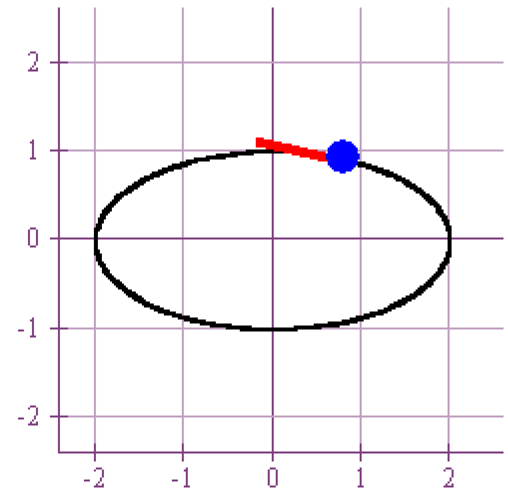
We don’t always have our curve parametrized by arc length.

$$curvature = \kappa = \left\| \frac{dT}{ds} \right\|$$



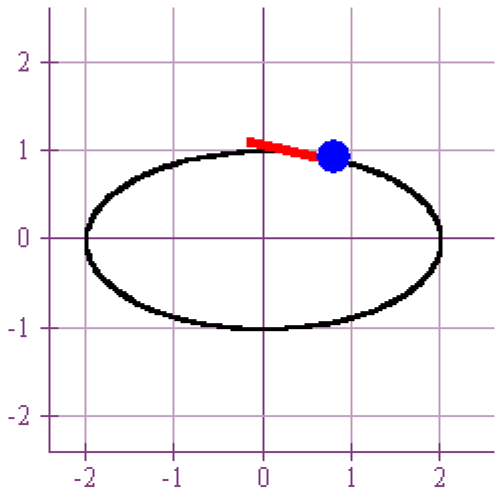
Here's where the chain rule comes to our rescue.

$$curvature = \kappa = \left\| \frac{dT}{ds} \right\|$$



By the chain rule, $\frac{dT}{dt} = \frac{dT}{ds} \cdot \frac{ds}{dt}$

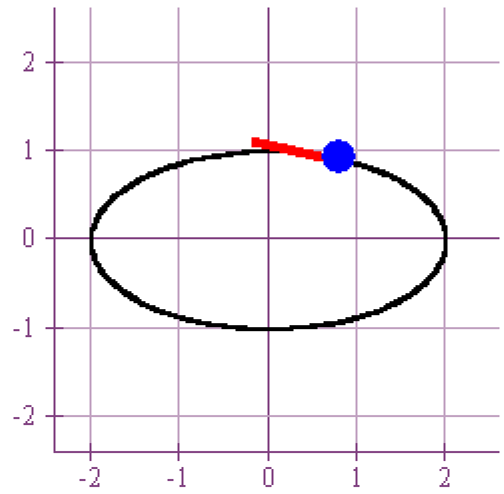
$$curvature = \kappa = \left\| \frac{dT}{ds} \right\|$$



By the chain rule, $\dfrac{dT}{dt} = \dfrac{dT}{ds} \cdot \dfrac{ds}{dt}$

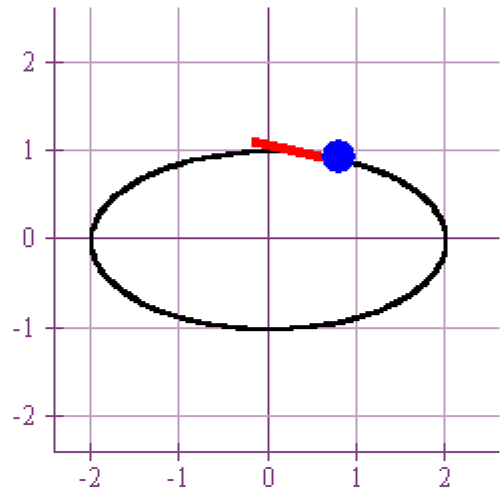
Recall, $\dfrac{ds}{dt} = \|\vec{v}(t)\| = \|\vec{r}\,'(t)\| = \left\| \dfrac{d\vec{r}}{dt} \right\|$

$$curvature = \kappa = \left\| \frac{dT}{ds} \right\|$$



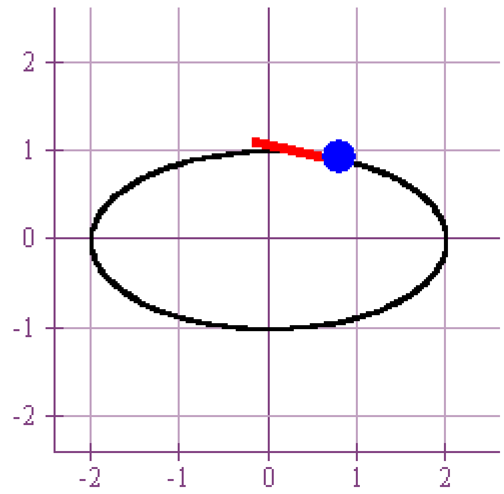
Hence, $\frac{dT}{dt} = \frac{dT}{ds} \cdot \frac{ds}{dt} = \frac{dT}{ds} \cdot \left\| \frac{d\vec{r}}{dt} \right\| \Rightarrow \frac{dT}{ds} = \frac{dT/dt}{\left\| d\vec{r}/dt \right\|}$

$$curvature = \kappa = \left\| \frac{dT}{ds} \right\|$$



Therefore,

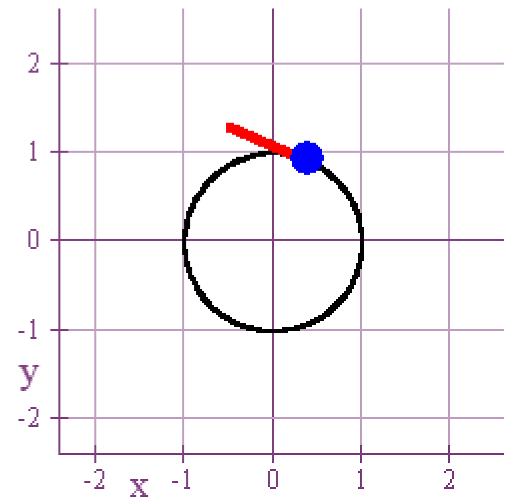
$$curvature = \kappa = \left\| \frac{dT}{ds} \right\| = \left\| \frac{dT/dt}{\|d\vec{r}/dt\|} \right\| = \frac{\|dT/dt\|}{\|d\vec{r}/dt\|} = \frac{\|T'(t)\|}{\|r'(t)\|}$$



Example: Below is a parametrization for a circle of radius *r* and center at the origin.

$$\vec{r}(t) = r\cos(t)\hat{i} + r\sin(t)\hat{j}$$

$$0 \le t < 2\pi$$

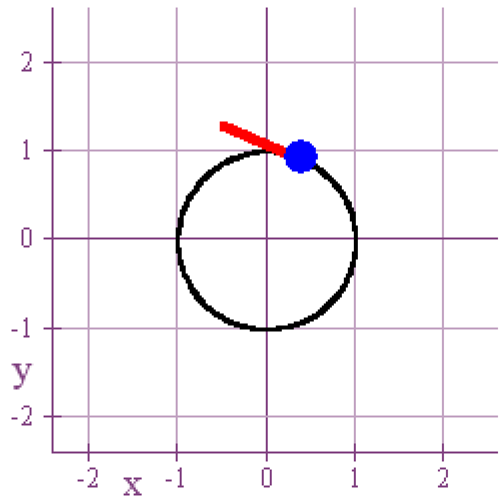


<u>Example:</u> First, find the unit tangent vector, T.

$$\vec{r}(t) = r\cos(t)\hat{i} + r\sin(t)\hat{j}$$

$$0 \le t < 2\pi$$

$$T = \frac{\vec{r}'(t)}{\left\|\vec{r}'(t)\right\|} = \frac{-r\sin(t)\hat{i} + r\cos(t)\hat{j}}{r} = -\sin(t)\hat{i} + \cos(t)\hat{j}$$



Example: Now find some derivatives.

$$T = -\sin(t)\hat{i} + \cos(t)\hat{j}$$

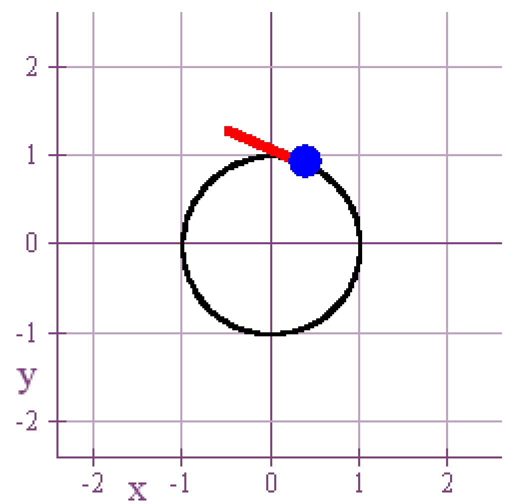
$$\vec{r}(t) = r\cos(t)\hat{i} + r\sin(t)\hat{j}$$

$$\frac{dT}{dt} = -\cos(t)\hat{i} - \sin(t)\hat{j}$$

$$\vec{r}'(t) = -r\sin(t)\hat{i} + r\cos(t)\hat{j}$$

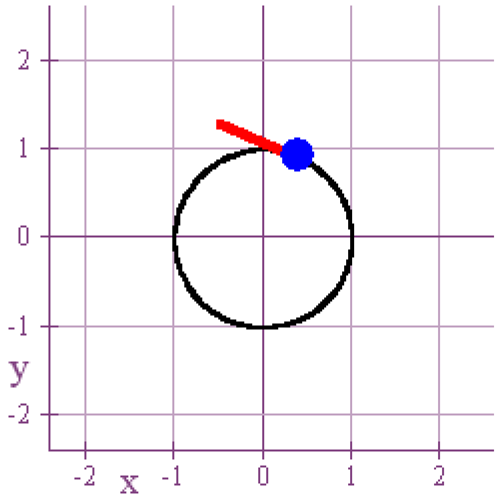
$$\left\| dT/dt \right\| = \left\| T'(t) \right\| = 1$$

$$\left\| d\vec{r}/dt \right\| = \left\| \vec{r}'(t) \right\| = r$$



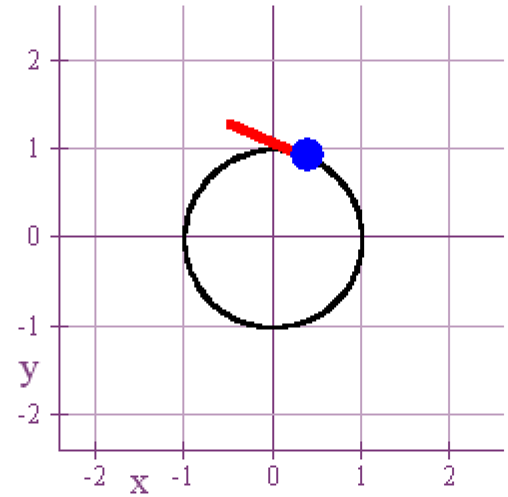
Example: Thus,

$$curvature = \kappa = \frac{\left\| T'(t) \right\|}{\left\| \vec{r}'(t) \right\|} = \frac{1}{r}$$



Example: In other words, we can think of a circle of radius *r* as having curvature *1/r* at every point.

$$curvature = \kappa = \frac{\left\| T'(t) \right\|}{\left\| \vec{r}'(t) \right\|} = \frac{1}{r}$$



Example: Apply this to the earth and explain why this makes good sense.

$$curvature = \kappa = \frac{\left\| T'(t) \right\|}{\left\| \vec{r}'(t) \right\|} = \frac{1}{r}$$