

# CONSTRAINED OPTIMIZATION



**EXAMPLE 1:** Suppose  $z = f(x, y) = x^2 + y^2 + 20$ , and our constraint curve is  $x + y = 3$ . Find the minimum value of  $z = f(x, y)$  on this curve.

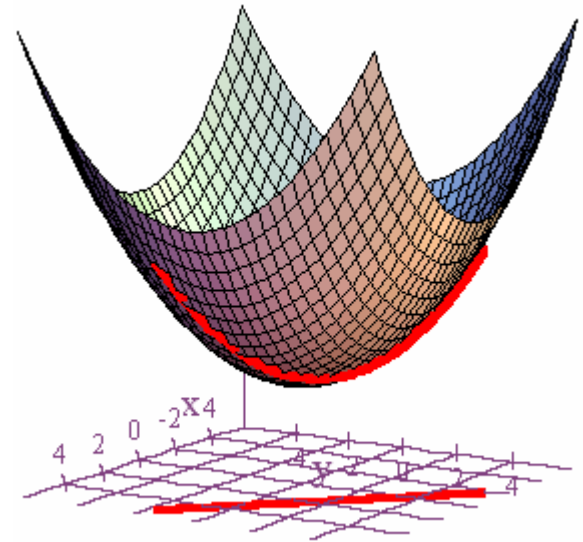
$$z = x^2 + y^2 + 20$$

$$g = x + y$$

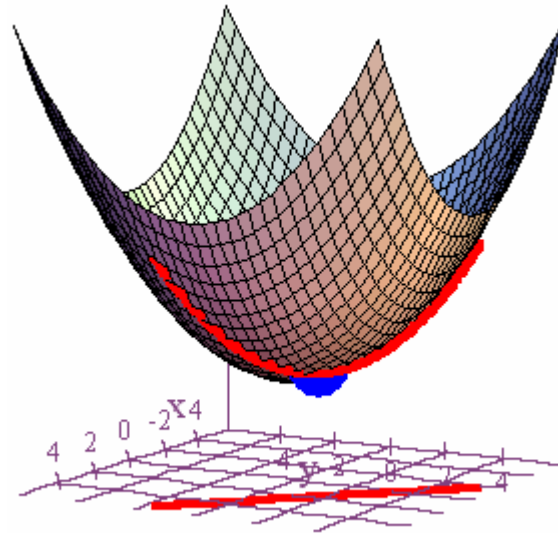
$$\begin{aligned} z_x = \lambda g_x &\Rightarrow 2x = \lambda &\Rightarrow x = \frac{\lambda}{2} \\ z_y = \lambda g_y &\Rightarrow 2y = \lambda &\Rightarrow y = \frac{\lambda}{2} \end{aligned}$$

$$x + y = 3 \Rightarrow \frac{\lambda}{2} + \frac{\lambda}{2} = 3 \Rightarrow \lambda = 3 \Rightarrow \begin{aligned} x &= 3/2 \\ y &= 3/2 \end{aligned}$$

$$f\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{9}{4} + \frac{9}{4} + 20 = \frac{49}{2} = 24.5$$



The minimum point is  $\left(\frac{3}{2}, \frac{3}{2}, \frac{49}{2}\right) = (1.5, 1.5, 24.5)$ .



EXAMPLE 2: Find the maximum and minimum values of

$$z = f(x, y) = x^2 + y^2 + 20 \text{ on the ellipse } \frac{x^2}{25} + \frac{y^2}{4} = 1.$$

$$z = x^2 + y^2 + 20$$

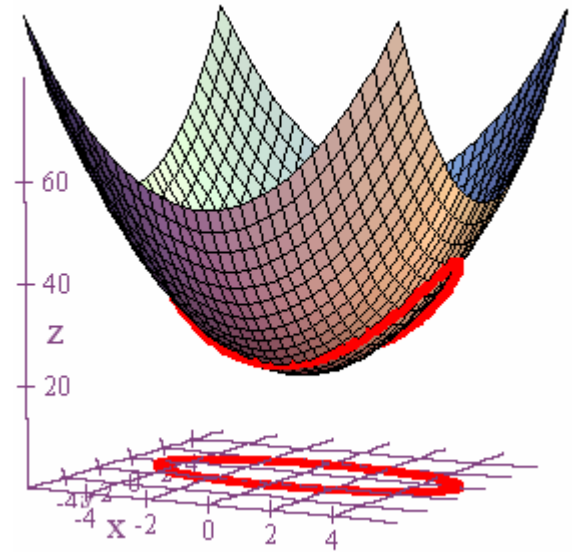
$$g = \frac{x^2}{25} + \frac{y^2}{4}$$

$$z_x = 2x$$

$$g_x = \frac{2x}{25} \Rightarrow 2x = \frac{2\lambda x}{25} \Rightarrow x = 0 \text{ or } \lambda = 25$$

$$z_y = 2y$$

$$g_y = \frac{2y}{4} \Rightarrow 2y = \frac{2\lambda y}{4} \Rightarrow y = 0 \text{ or } \lambda = 4$$



$$z = x^2 + y^2 + 20$$

$$z_x = 2x$$

$$g_x = \frac{2x}{25} \Rightarrow 2x = \frac{2\lambda x}{25} \Rightarrow x = 0 \text{ or } \lambda = 25$$

$$g = \frac{x^2}{25} + \frac{y^2}{4}$$

$$z_y = 2y$$

$$g_y = \frac{2y}{4} \Rightarrow 2y = \frac{2\lambda y}{4} \Rightarrow y = 0 \text{ or } \lambda = 4$$

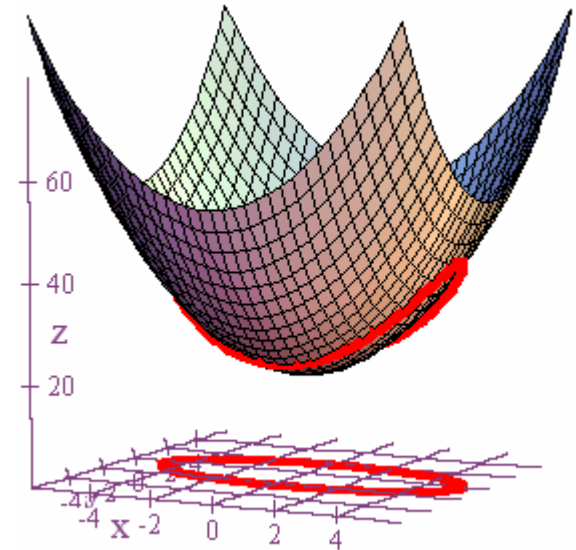
Hence, extreme values can only occur at the points  $(0, -2)$ ,  $(0, 2)$ ,  $(-5, 0)$ , and  $(5, 0)$ .

$$z(0, -2) = 24 = \min$$

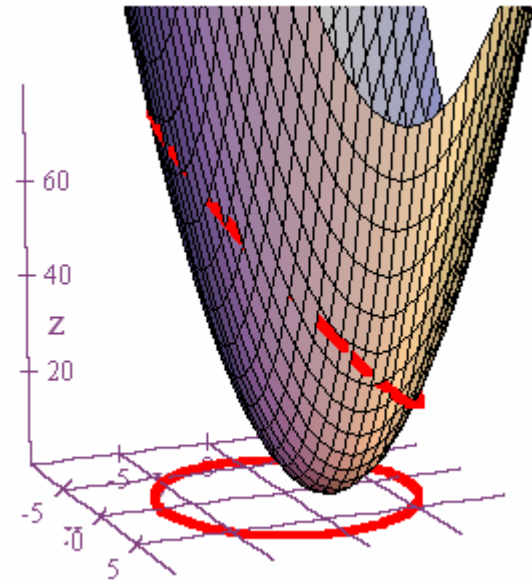
$$z(0, 2) = 24 = \min$$

$$z(-5, 0) = 45 = \max$$

$$z(5, 0) = 45 = \max$$



EXAMPLE 3: Find the maximum and minimum values of  $z = f(x, y) = (x - 1)^2 + (y - 2)^2$  on the circle  $x^2 + y^2 = 45$ .



EXAMPLE 3: Find the maximum and minimum values of

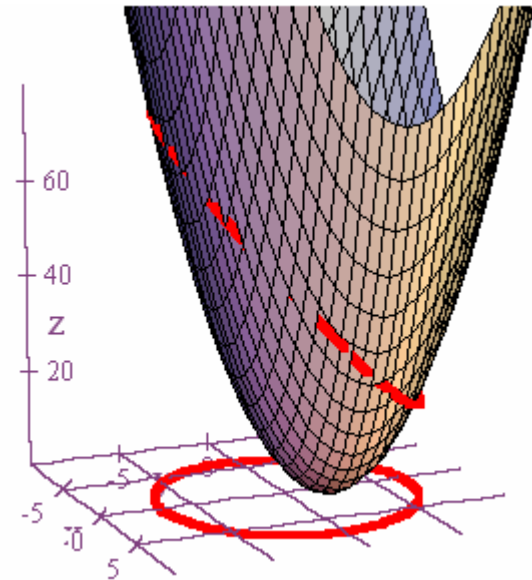
$z = f(x, y) = (x-1)^2 + (y-2)^2$  on the circle  $x^2 + y^2 = 45$ .

$$z = (x-1)^2 + (y-2)^2$$

$$g = x^2 + y^2$$

$$\begin{aligned} z_x &= 2x - 2 \\ g_x &= 2x \end{aligned} \Rightarrow 2x - 2 = 2\lambda x \Rightarrow x - 1 = \lambda x$$

$$\begin{aligned} z_y &= 2y - 4 \\ g_y &= 2y \end{aligned} \Rightarrow 2y - 4 = 2\lambda y \Rightarrow y - 2 = \lambda y$$



$$x - 1 = \lambda x \Rightarrow x - \lambda x = 1 \Rightarrow x(1 - \lambda) = 1 \Rightarrow x = \frac{1}{1 - \lambda}$$

$$z = (x - 1)^2 + (y - 2)^2$$
$$g = x^2 + y^2$$

$$y - 2 = \lambda y \Rightarrow y - \lambda y = 2 \Rightarrow y(1 - \lambda) = 2 \Rightarrow y = \frac{2}{1 - \lambda}$$

$$x^2 + y^2 = 45 \Rightarrow \left(\frac{1}{1 - \lambda}\right)^2 + \left(\frac{2}{1 - \lambda}\right)^2 = 45$$

$$\Rightarrow \frac{5}{(1 - \lambda)^2} = 45 \Rightarrow (1 - \lambda)^2 = \frac{1}{9}$$

$$\Rightarrow \lambda^2 - 2\lambda + \frac{8}{9} = 0 \Rightarrow \lambda = \begin{cases} 4/3 \\ 2/3 \end{cases}$$

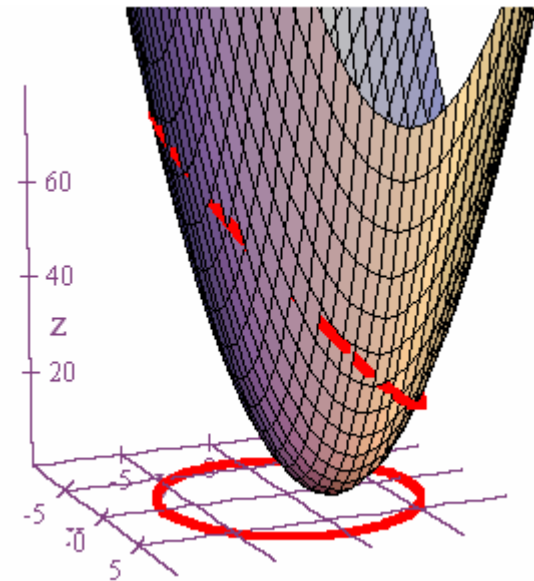
$$\Rightarrow x = \pm 3 \text{ and } y = \pm 6$$

$$z(-3, -6) = (-4)^2 + (-8)^2 = 16 + 64 = 80 \leftarrow \text{maximum}$$

$$z(-3, 6) = (-4)^2 + 4^2 = 16 + 16 = 32$$

$$z(3, -6) = 2^2 + (-8)^2 = 4 + 64 = 68$$

$$z(3, 6) = 2^2 + 4^2 = 4 + 16 = 20 \leftarrow \text{minimum}$$





$$x - 1 = \lambda x \Rightarrow x - \lambda x = 1 \Rightarrow x(1 - \lambda) = 1 \Rightarrow x = \frac{1}{1 - \lambda}$$

$$z = (x - 1)^2 + (y - 2)^2$$
$$g = x^2 + y^2$$

$$y - 2 = \lambda y \Rightarrow y - \lambda y = 2 \Rightarrow y(1 - \lambda) = 2 \Rightarrow y = \frac{2}{1 - \lambda}$$

$$x^2 + y^2 = 45 \Rightarrow \left(\frac{1}{1 - \lambda}\right)^2 + \left(\frac{2}{1 - \lambda}\right)^2 = 45$$

$$\Rightarrow \frac{5}{(1 - \lambda)^2} = 45 \Rightarrow (1 - \lambda)^2 = \frac{1}{9}$$

$$\Rightarrow \lambda^2 - 2\lambda + \frac{8}{9} = 0 \Rightarrow \lambda = \begin{cases} 4/3 \\ 2/3 \end{cases}$$

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$$z(-3, 6) = (-4)^2 + 4^2 = 16 + 16 = 32$$

$$z(3, -6) = 2^2 + (-8)^2 = 4 + 64 = 68$$

$$z(3, 6) = 2^2 + 4^2 = 4 + 16 = 20 \leftarrow \text{minimum}$$

