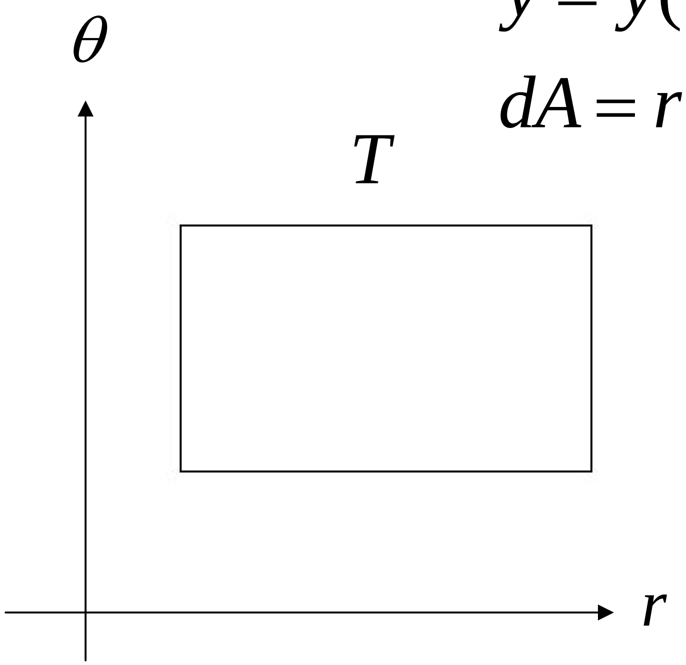
Change of Variables In Multiple Integrals

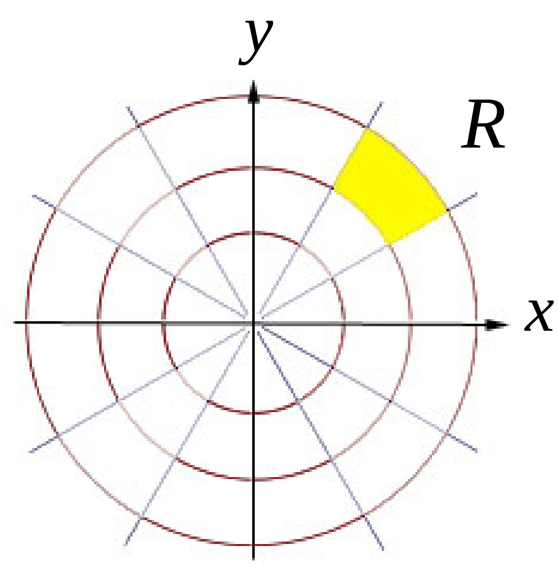
When we convert a double integral from rectangular to polar coordinates, recall the changes that must be made to *x*, *y and dA*.

$$x = x(r,\theta) = r\cos\theta$$

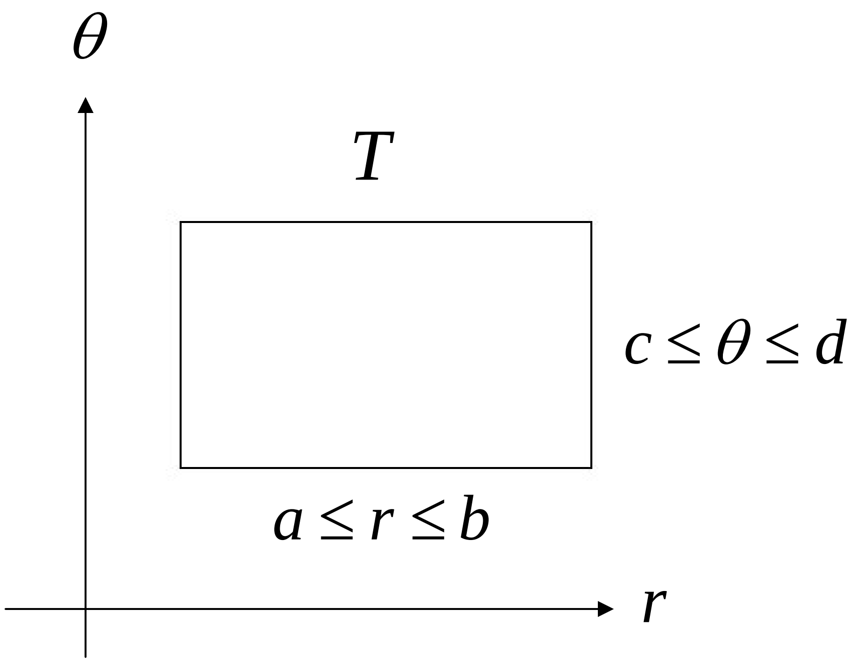
$$y = y(r,\theta) = r\sin\theta$$

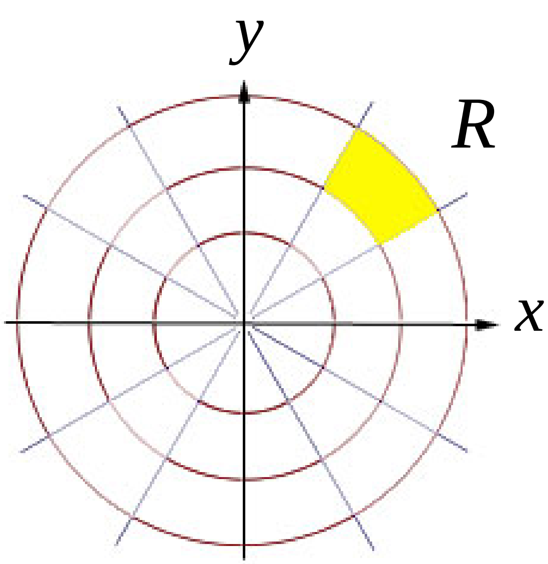
$$dA = r\,dr\,d\theta$$





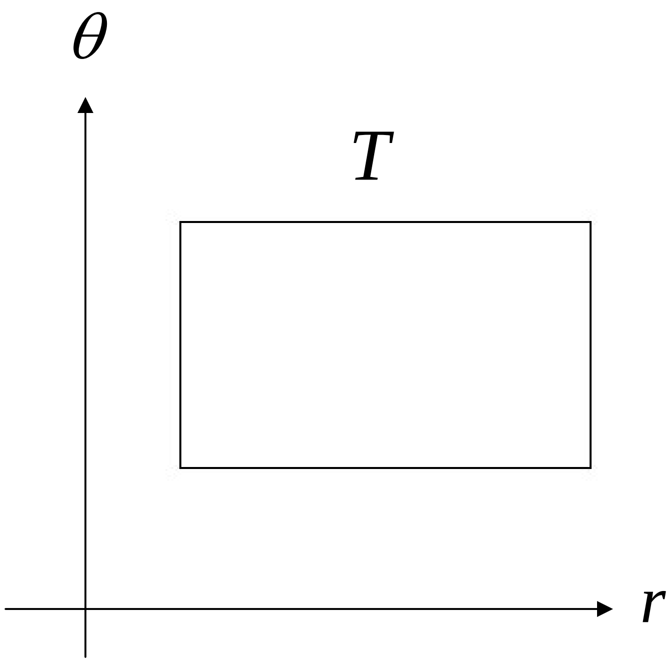
In the polar coordinate system, an element of area is generally a rectangle corresponding to a range of values for *r* and a range for *θ*.

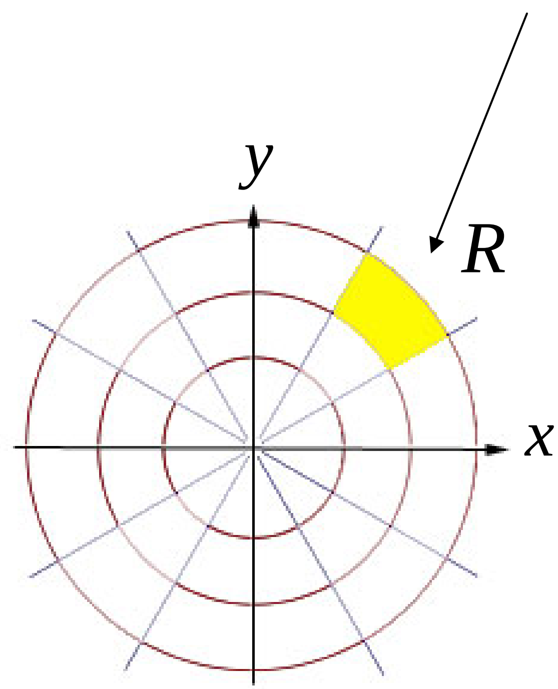




However, in the *xy* coordinate system, this rectangle usually takes on a different shape, and the formula for an element of area changes.

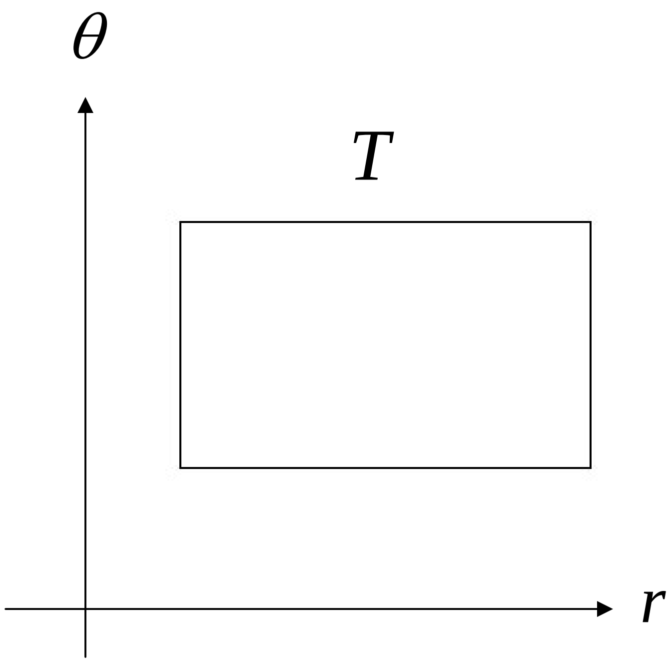
$$\Delta A \approx r \cdot \Delta r \cdot \Delta\theta$$

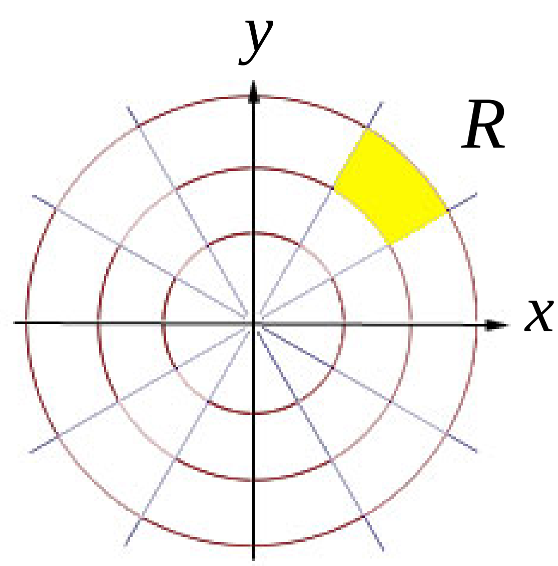




This leads to the following formula for the double integral in polar coordinates.

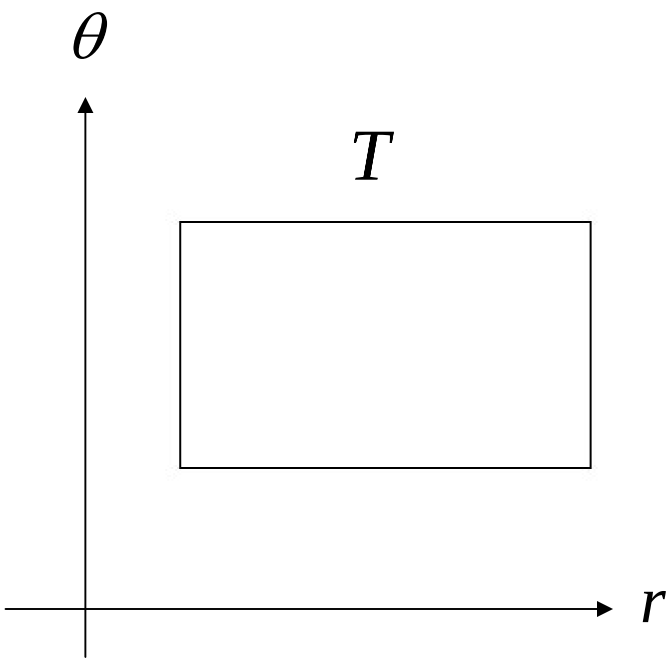
$$\iint_R f(x,y)\,dA = \iint_T f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

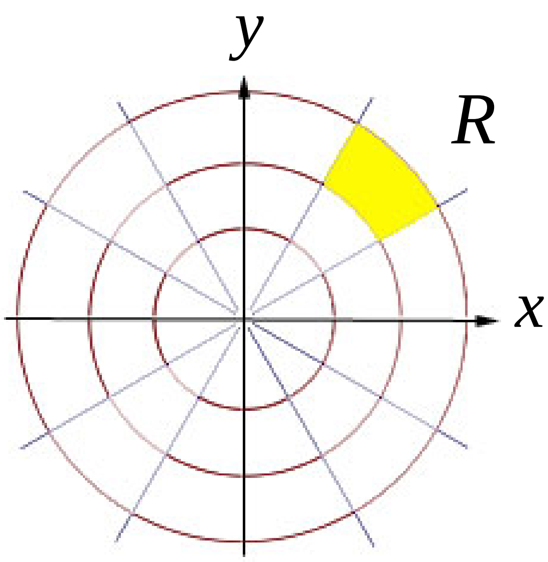




We'll now develop a general method for finding change of variable formulas such as this one.

$$\iint_R f(x,y)\,dA = \iint_T f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

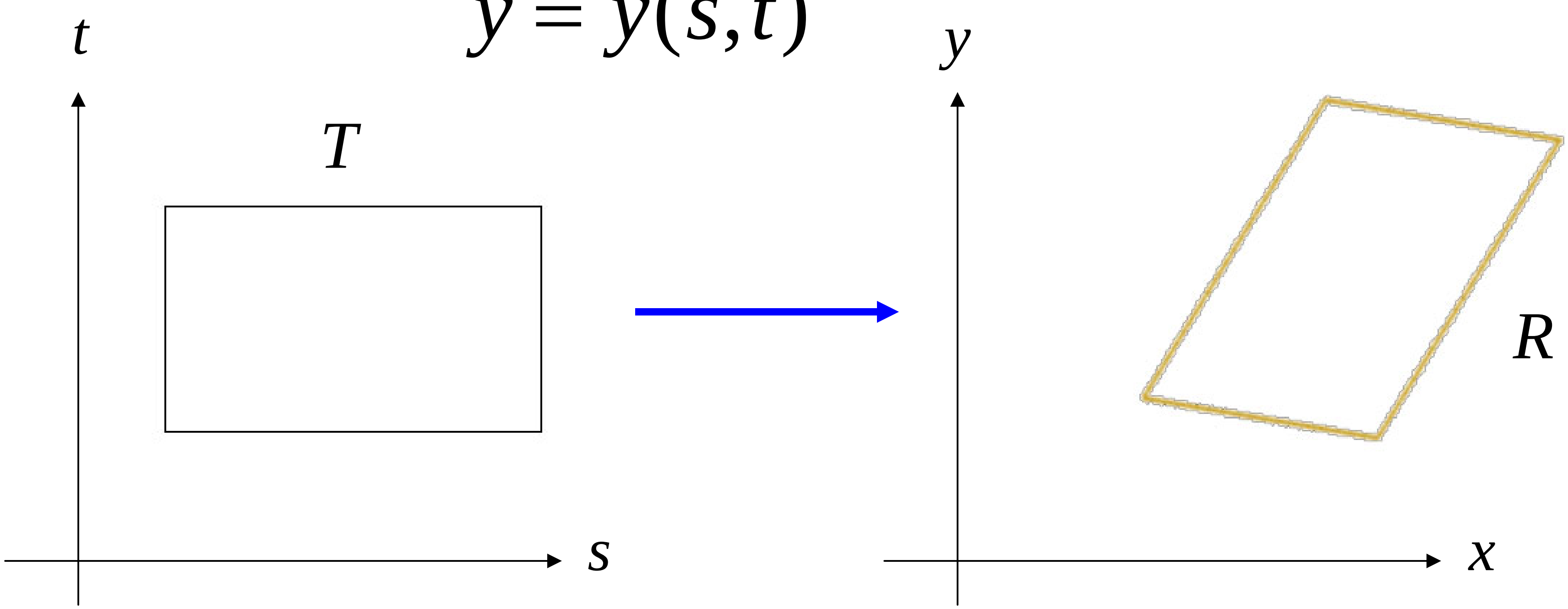




Suppose we have a rectangle in an *st*-coordinate system and a pair of functions that converts *(s,t)* coordinates into *(x,y)* coordinates.

$$x = x(s,t)$$

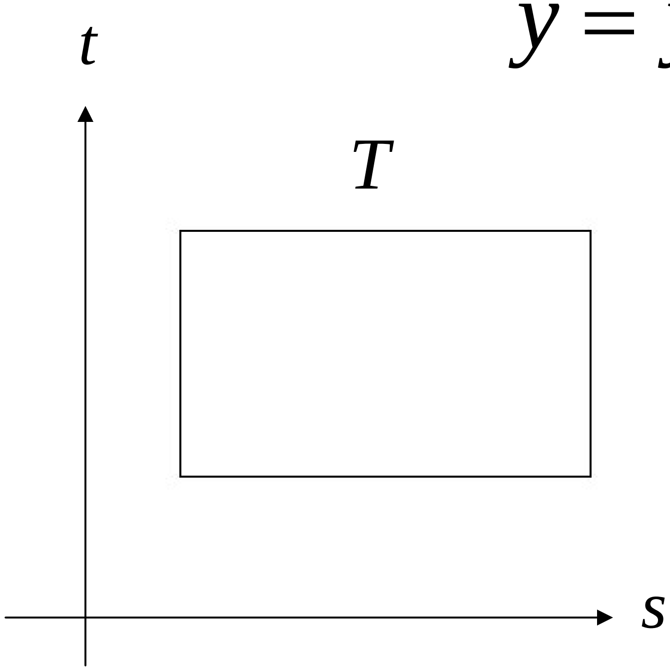
$$y = y(s,t)$$

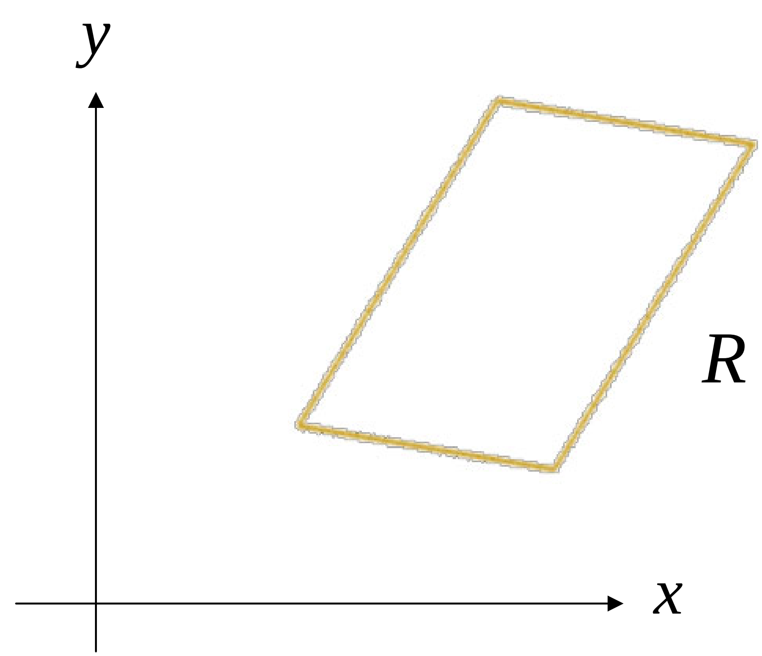


Suppose also that these functions are differentiable and that the transformation from the *st*-coordinates to *xy*-coordinates is one-to-one.

$$x = x(s,t)$$

$$y = y(s,t)$$

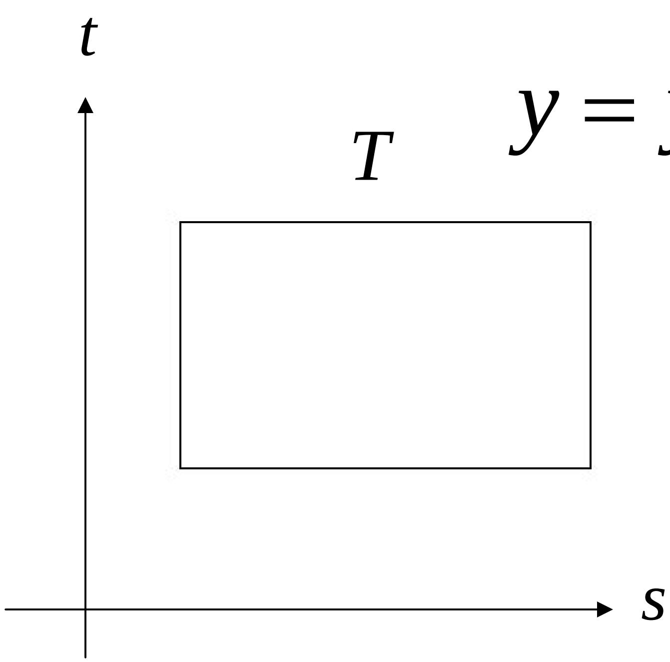


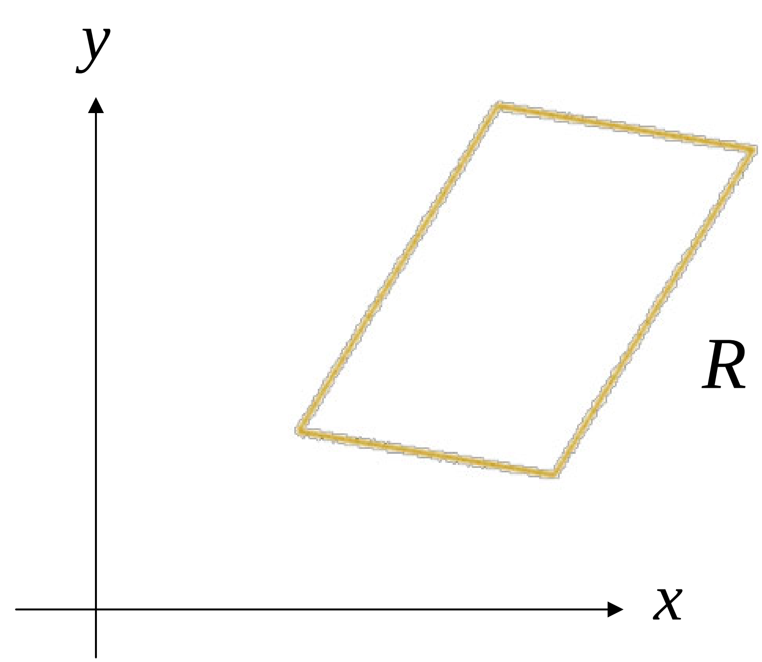


Then because of differentiability, local linearity will be present and a small rectangle in the *st* system will be mapped onto approximately a parallelogram in the *xy*-coordinate system.

$$x = x(s,t)$$

$$y = y(s,t)$$

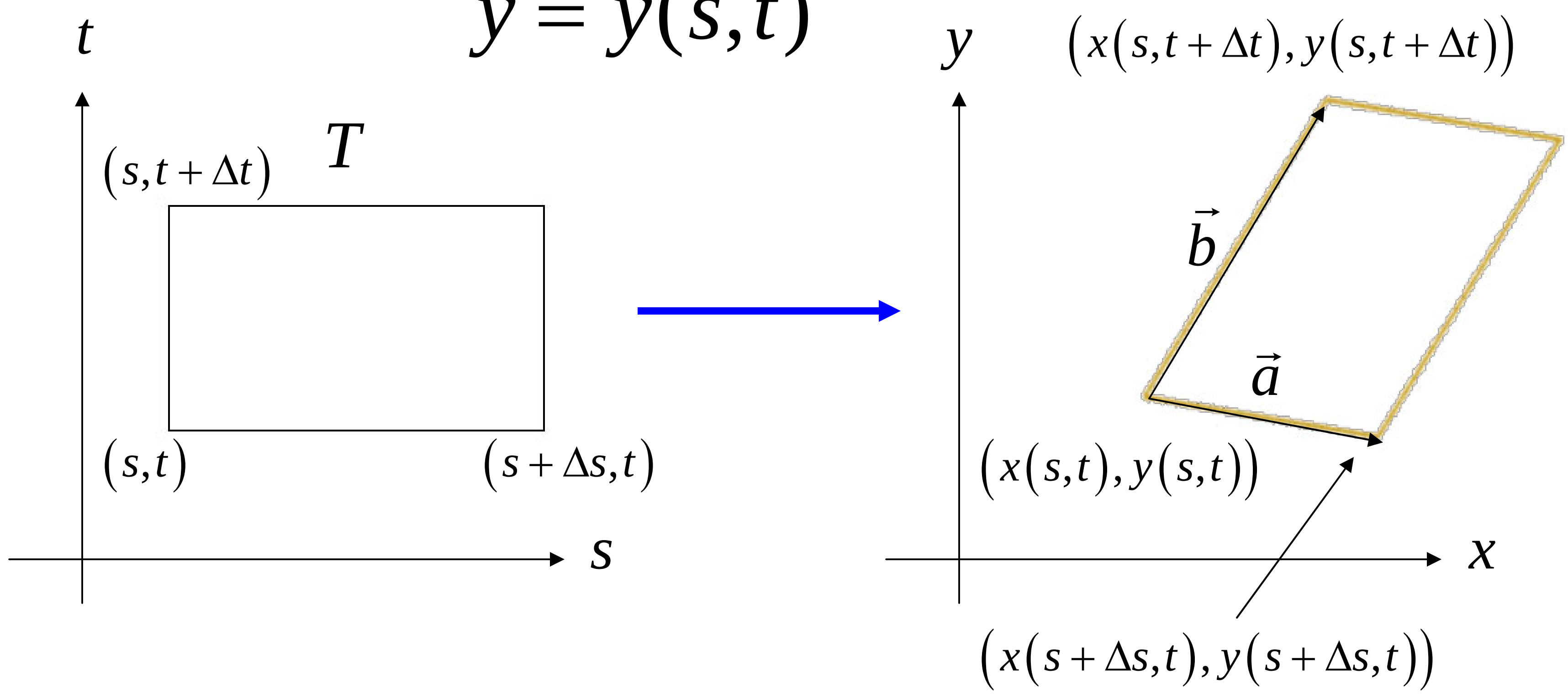




If we add some coordinates, then it looks like this.

$$x = x(s,t)$$

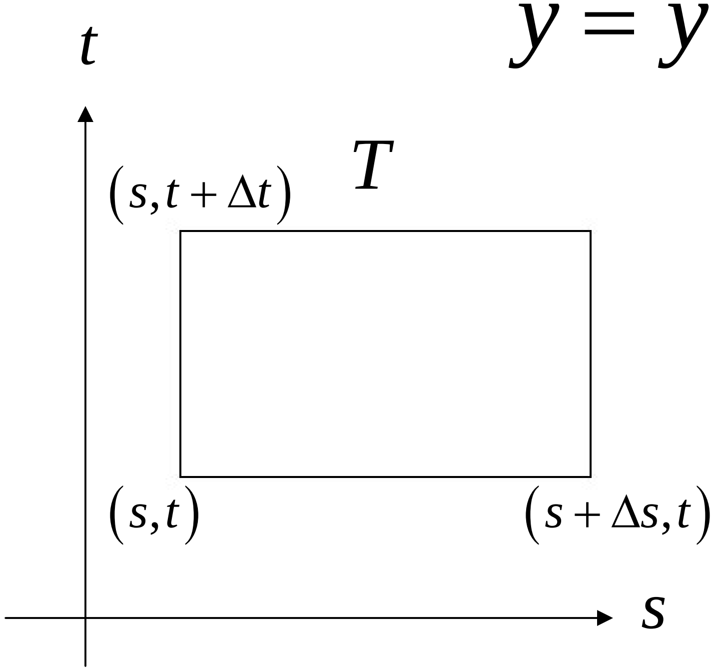
$$y = y(s,t)$$

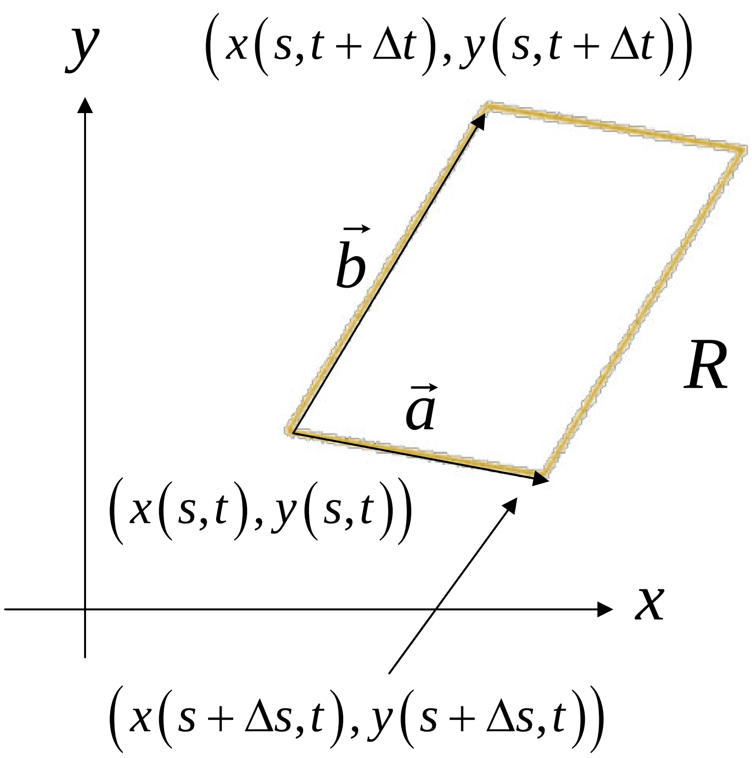


An element of area in our *xy*-coordinate system is represented by a parallelogram.

$$x = x(s,t)$$

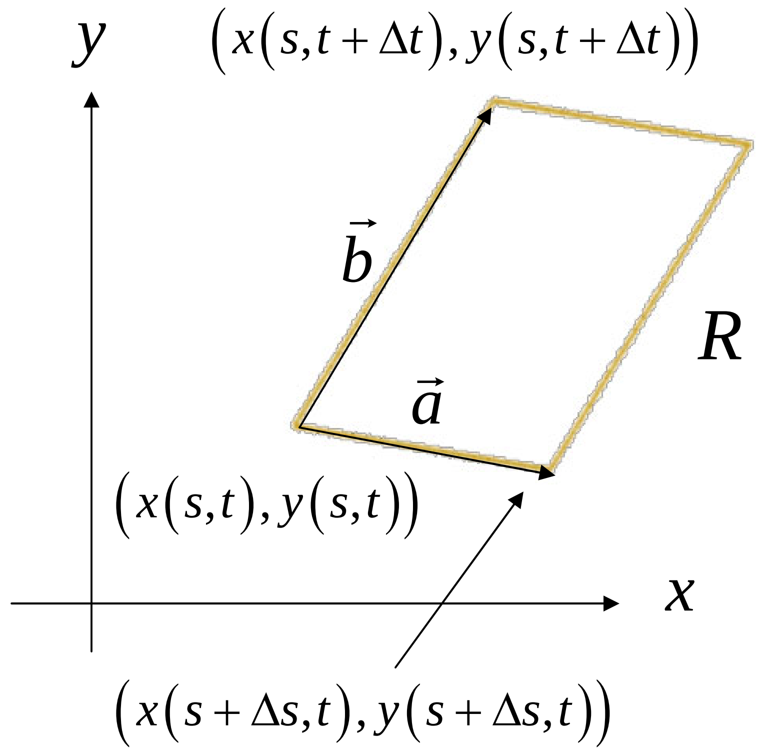
$$y = y(s,t)$$





The area of this parallelogram is given by the norm of a cross product.

$$Area = \left\| \vec{a} \times \vec{b} \right\|$$



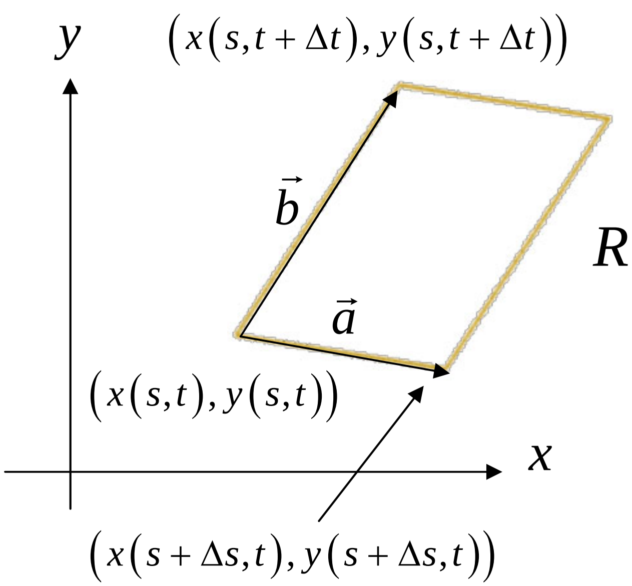
Notice that,

$$\vec{a} = \left(x\left(s+\Delta s,t\right) - x\left(s,t\right)\right)\hat{i} + \left(y\left(s+\Delta s,t\right) - y\left(s,t\right)\right)\hat{j}$$

$$\approx \frac{\partial x}{\partial s}\Delta s\,\hat{i} + \frac{\partial y}{\partial s}\Delta s\,\hat{j}$$

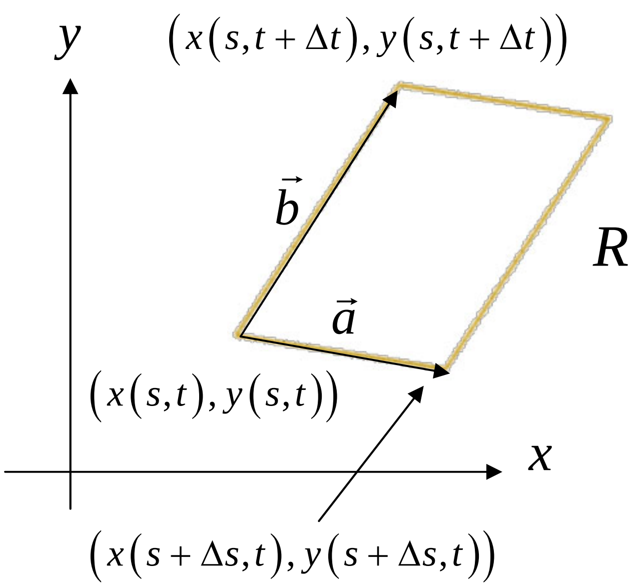
$$\vec{b} = \left(x\left(s,t+\Delta t\right) - x\left(s,t\right)\right)\hat{i} + \left(y\left(s,t+\Delta t\right) - y\left(s,t\right)\right)\hat{j}$$

$$\approx \frac{\partial x}{\partial t}\Delta t\,\hat{i} + \frac{\partial y}{\partial t}\Delta t\,\hat{j}$$



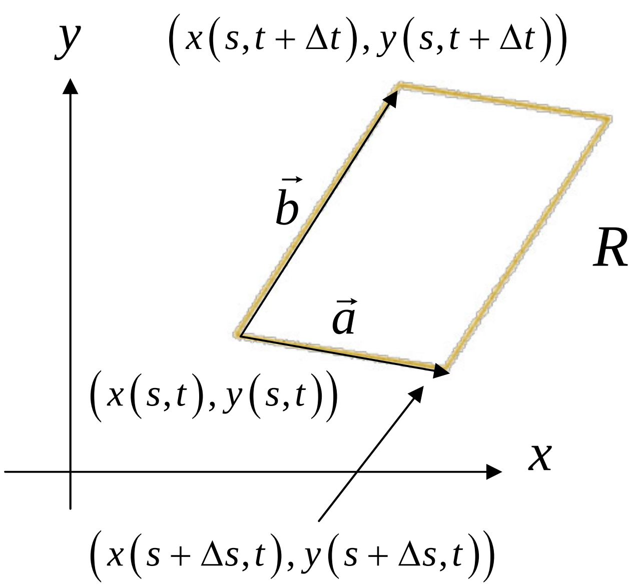
Hence,

$$\vec{a}\times\vec{b}\approx\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial x}{\partial s}\Delta s & \dfrac{\partial y}{\partial s}\Delta s & 0\\ \dfrac{\partial x}{\partial t}\Delta t & \dfrac{\partial y}{\partial t}\Delta t & 0\end{vmatrix}=\left(\frac{\partial x}{\partial s}\Delta s\cdot\frac{\partial y}{\partial t}\Delta t-\frac{\partial y}{\partial s}\Delta s\cdot\frac{\partial x}{\partial t}\Delta t\right)\hat{k}$$



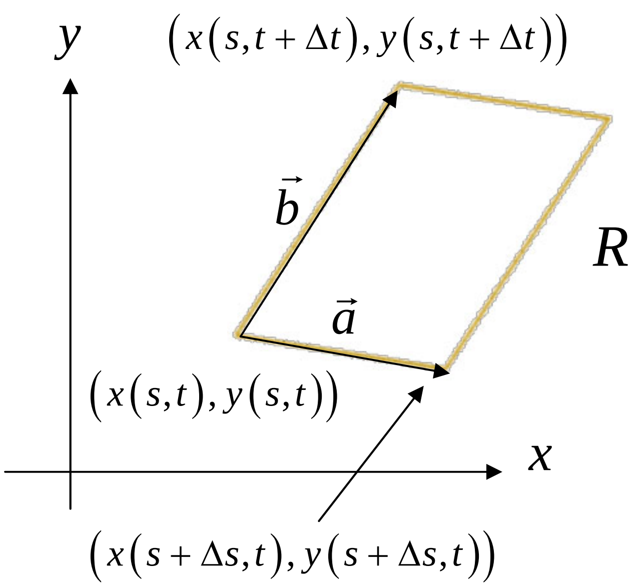
And,

$$\left\|\vec{a}\times\vec{b}\right\| \approx \left|\frac{\partial x}{\partial s}\Delta s\cdot\frac{\partial y}{\partial t}\Delta t-\frac{\partial y}{\partial s}\Delta s\cdot\frac{\partial x}{\partial t}\Delta t\right| = \left|\frac{\partial x}{\partial s}\frac{\partial y}{\partial t}-\frac{\partial y}{\partial s}\frac{\partial x}{\partial t}\right|\Delta s\Delta t$$



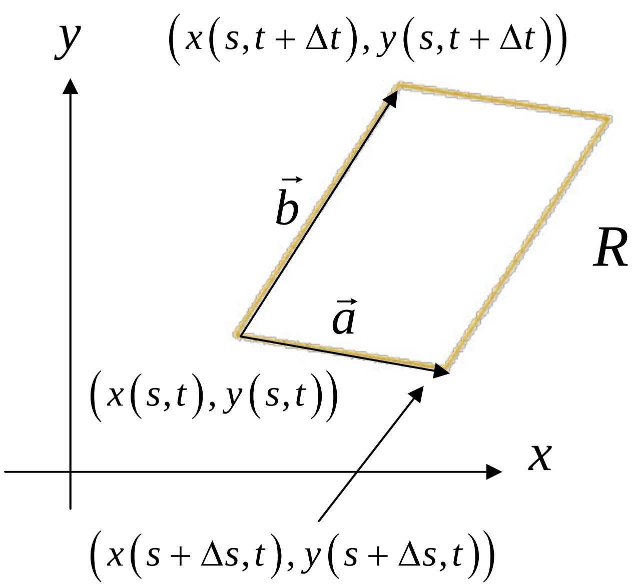
The expression inside the last absolute value sign is called the *Jacobian*, and it is usually written as follows:

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{vmatrix} = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial y}{\partial s}\frac{\partial x}{\partial t}$$



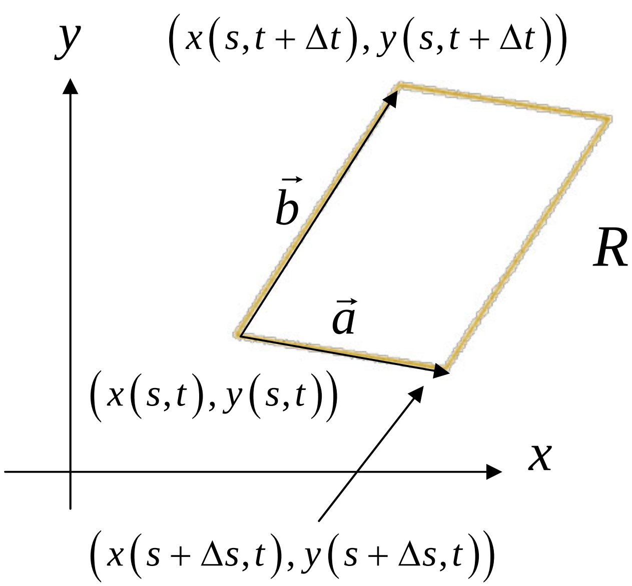
Therefore,

$$Area = \Delta A = \left\| \vec{a} \times \vec{b} \right\| \approx \left| \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial y}{\partial s}\frac{\partial x}{\partial t} \right| \Delta s \Delta t = \left| \frac{\partial (x,y)}{\partial (s,t)} \right| \Delta s \Delta t$$



And as a result of local linearity, this approximation improves as the changes in *s* and *t* become smaller.

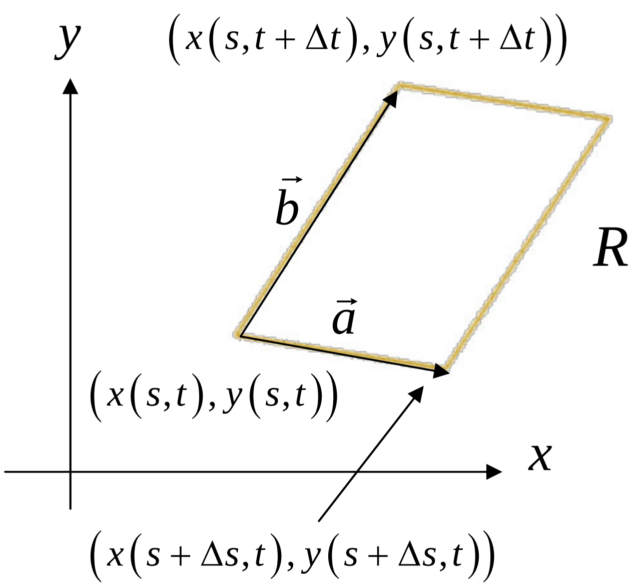
$$Area = \Delta A = \left\| \vec{a} \times \vec{b} \right\| \approx \left| \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial y}{\partial s}\frac{\partial x}{\partial t} \right| \Delta s \Delta t = \left| \frac{\partial (x,y)}{\partial (s,t)} \right| \Delta s \Delta t$$



And finally,

$$\iint_R f(x,y)\,dA = \lim_{\Delta A \to 0} \sum f(x,y) \cdot \Delta A$$

$$= \lim_{\Delta s, \Delta t \to 0} \sum f(x(s,t), y(s,t)) \cdot \left| \frac{\partial(x,y)}{\partial(s,t)} \right| \Delta s \Delta t$$

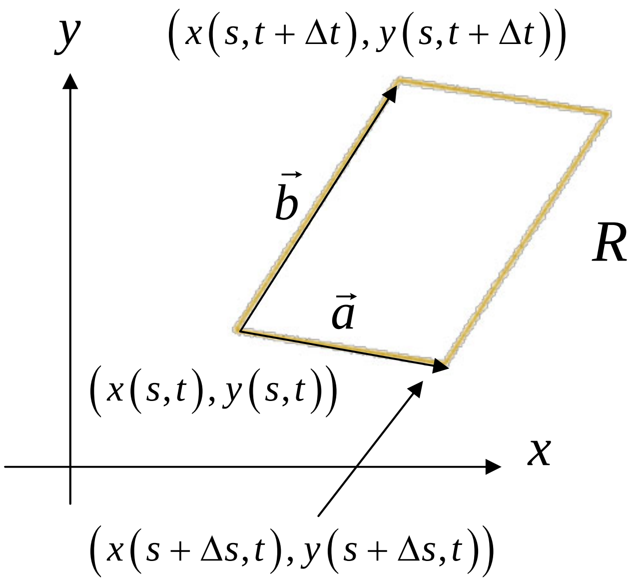


$$= \iint_T f(x(s,t), y(s,t)) \cdot \left| \frac{\partial(x,y)}{\partial(s,t)} \right| ds\,dt$$

Now let's verify that this formula works for polar coordinates.

$$\iint_R f(x,y)\,dA = \lim_{\Delta A \to 0} \sum f(x,y)\cdot \Delta A$$

$$= \lim_{\Delta s, \Delta t \to 0} \sum f(x,y)\cdot \left|\frac{\partial(x,y)}{\partial(s,t)}\right| \Delta s \Delta t = \iint_T f(x,y)\cdot \left|\frac{\partial(x,y)}{\partial(s,t)}\right| ds\,dt$$



$$\int\int_R f(x,y)\,dA = \int\int_T f(x(s,t),y(s,t)) \cdot \left|\frac{\partial\left(x,y\right)}{\partial\left(r,\theta\right)}\right| dr\,d\theta$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r\left(\cos^2\theta + \sin^2\theta\right) = r$$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)} = \right| = |r| = r$$

Therefore,

$$\iint_R f(x,y)\,dA = \iint_T f(r\cos\theta, r\sin\theta)\cdot\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$= \iint_T f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta$$

Let’s try one more example! Consider the equation below for an ellipse.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

The following substitution will change it into an equation for a unit circle.

$$x = a \cdot s$$

$$y = b \cdot t$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{(a \cdot s)^2}{a^2} + \frac{(b \cdot t)^2}{b^2} = 1 \Rightarrow s^2 + t^2 = 1$$

Now find the absolute value of the *Jacobian*.

$$x = a \cdot s$$

$$y = b \cdot t$$

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

$$\left| \frac{\partial(x,y)}{\partial(s,t)} \right| = ab$$

We can now easily find the area of the ellipse.

$$Area\,of\;ellipse = \iint_{ellipse} dA = \iint_{unit\,circle} ab\,dsdt$$

$$= ab\left(\iint_{unit\,circle} dsdt\right) = ab \cdot \pi = \pi ab$$

If we have a function of three variables, then our Jacobian looks like this.

$$\frac{\partial(x,y,z)}{\partial(s,t,u)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} & \frac{\partial z}{\partial u} \end{vmatrix}$$

Find the volume of the ellipsoid.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

The coordinate change below transforms the ellipsoid into a unit sphere.

$$x = a \cdot s$$

$$y = b \cdot t$$

$$z = c \cdot u$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{\partial(x,y,z)}{\partial(s,t,u)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} & \frac{\partial z}{\partial u} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$Volume=\iiint_R dV=\iiint_T \left|\frac{\partial(x,y,z)}{\partial(s,t,u)}\right| dsdtdu$$

$$=\iiint_T abc\, dsdtdu = abc\iiint_T dsdtdu$$

$$=abc\cdot\frac{4\pi}{3}=\frac{4}{3}\pi abc$$

Any Questions?